

## The Complexity of Simultaneous Geometric Graph Embedding

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### Abstract

Given a collection of planar graphs  $G_1, \dots, G_k$  on the same set  $V$  of  $n$  vertices, the *simultaneous geometric embedding (with mapping)* problem, or simply  $k$ -SGE, is to find a set  $P$  of  $n$  points in the plane and a bijection  $\varphi : V \rightarrow P$  such that the induced straight-line drawings of  $G_1, \dots, G_k$  under  $\varphi$  are all plane.

This problem is polynomial-time equivalent to weak rectilinear realizability of abstract topological graphs, which Kynčl (doi:10.1007/s00454-010-9320-x) proved to be complete for  $\exists\mathbb{R}$ , the existential theory of the reals. Hence the problem  $k$ -SGE is polynomial-time equivalent to several other problems in computational geometry, such as recognizing intersection graphs of line segments or finding the rectilinear crossing number of a graph.

We give an elementary reduction from the pseudoline stretchability problem to  $k$ -SGE, with the property that both numbers  $k$  and  $n$  are linear in the number of pseudolines. This implies not only the  $\exists\mathbb{R}$ -hardness result, but also a  $2^{2^{\Omega(n)}}$  lower bound on the minimum size of a grid on which any such simultaneous embedding can be drawn. This bound is tight. Hence there exists such collections of graphs that can be simultaneously embedded, but every simultaneous drawing requires an exponential number of bits per coordinates. The best value that can be extracted from Kynčl's proof is only  $2^{2^{\Omega(\sqrt{n})}}$ .

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## 1 Introduction

Given graphs  $G_1 = (V, E_1), \dots, G_k = (V, E_k)$  on  $n$  vertices, *simultaneous geometric embedding (with mapping)* or simply  $k$ -SGE, is the problem of finding a point set  $P \subset \mathbb{R}^2$  of size  $n$  and a bijection  $\varphi : V \rightarrow P$  such that the induced straight-line drawings of  $G_1, \dots, G_k$  under  $\varphi$  are all plane [7]. The corresponding decision problem (which we also refer to as  $k$ -SGE) simply asks whether such a point set exists. It is important to note that  $k$  is part of the input and can thus depend on  $n$ . The problem 1-SGE amounts to planarity testing. The problem 2-SGE is typically referred to simply as SGE. Fig. 1 shows an example of two graphs and a 2-SGE.

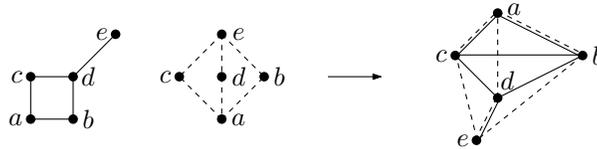


Figure 1: Graphs  $G_1$  and  $G_2$  on the same vertex set and a 2-SGE of  $G_1$  and  $G_2$ .

Early work on the topic focused on the existence of  $k$ -SGEs for restricted graph classes. The SGE problem was originally introduced by Brass et. al [7]. They show that there is a pair of outerplanar graphs on the same vertex set that does not admit a 2-SGE. Additionally, they give a triple of paths that does not admit a 3-SGE. The authors also show that various other classes of graphs, such as a pair of caterpillars, an extended star and a path, or two stars always admit a 2-SGE. The most recent positive result is a 2-SGE construction that works for generalizations of caterpillars with generalizations of stars, spiders and caterpillars [9]. The question of whether any two trees admit a 2-SGE remained open for six years, until the question was settled in the negative with a counterexample [15]. The most recent negative result gives a tree and a path that do not admit a 2-SGE [2]. Research has since focused on other variations of the problem, such as *simultaneous embedding with fixed edges* (edges are drawn as arbitrary simple curves, but all graphs must use identical curves for identical edges), *matched drawings* (vertices have fixed  $y$ -coordinates in all drawings, but may have different  $x$ -coordinates in each drawing), or *partial simultaneous geometric embedding* (a limited number of vertices may be mapped to different points in different drawings) [11]. The decision problem 2-SGE is NP-hard [10]. See [6] for an excellent survey.

The existential theory of the reals is the set of true sentences of the form  $\exists(x_1, \dots, x_n) : \varphi(x_1, \dots, x_n)$ , where  $\varphi$  is a  $(\wedge, \vee, \neg)$ -formula over the signature  $(0, 1, +, *, <, \leq, =)$  interpreted over the universe of real numbers [30]. The decision problem ETR asks whether a given sentence is true. The complexity class  $\exists\mathbb{R}$  is defined as the set of decision problems that can be reduced to ETR in polynomial time. A problem is  $\exists\mathbb{R}$ -hard if it is at least as hard as every problem in  $\exists\mathbb{R}$ , i.e., if every problem in  $\exists\mathbb{R}$  can be reduced to it in polynomial

time [29, 3]. A problem is  $\exists\mathbb{R}$ -complete if it belongs to  $\exists\mathbb{R}$  and is  $\exists\mathbb{R}$ -hard. It is known that  $\text{NP} \subseteq \exists\mathbb{R} \subseteq \text{PSPACE}$ : Boolean satisfiability can be encoded as a decision problem on a set of polynomial inequalities and Canny [8] gave a polynomial-space algorithm for ETR.

In 2011, Kynčl [22] proved that *weak rectilinear realizability of abstract topological graphs* is  $\exists\mathbb{R}$ -complete. Since this problem reduces to  $k$ -SGE in polynomial time [14] and since  $k$ -SGE belongs to  $\exists\mathbb{R}$  [10], it follows that  $k$ -SGE is  $\exists\mathbb{R}$ -complete. The  $k$ -SGE problem is therefore polynomial-time equivalent to many other classical problems in computational geometry, such as finding the rectilinear crossing number of a graph [4], recognizing unit disk graphs [25], recognizing intersection graphs of convex sets in the plane [29], recognizing intersection graphs of segments [21, 24], solving the Steinitz problem [26], and deciding the realizability of linkages [19]. We refer the reader to recent work of Schaefer for more references and examples [29, 30].

Our contribution is an elementary self-contained construction showing the  $\exists\mathbb{R}$ -hardness of  $k$ -SGE. It involves a direct translation of the information contained in an arrangement of  $n$  pseudolines into a set of  $n$  planar graphs on a set  $V$  of  $O(n)$  vertices, in such a way that the pseudoline arrangement is stretchable if and only if the graphs can be simultaneously embedded. The main interesting feature of this construction is that the size of  $V$  is linear in the number of pseudolines. This implies that for some positive instances of  $k$ -SGE, representing the point set by encoding the coordinates of each point requires an *exponential number of bits*. This follows from the analogous result on realizations of order types by Goodman, Pollack, and Sturmfels [17]. Our result improves on Kynčl’s construction, which shows only that  $2^{\Omega(\sqrt{n})}$  bits are sometimes necessary.

In Section 2, we briefly recall standard results on (realizability of) order types and (stretchability of) pseudoline arrangements. The reduction itself is given in Section 3. Section 4 presents our results on coordinate sizes in simultaneous embeddings.

## 2 Pseudolines and order types

Many combinatorial properties of a point set in the plane are captured by its *order type*. The order type of a point set  $P \subset \mathbb{R}^2$  is the mapping  $\chi : \binom{P}{3} \rightarrow \{-1, 0, +1\}$ , where

$$\chi(a, b, c) = \text{sign} \left( \begin{vmatrix} a_x & a_y & 1 \\ b_x & b_y & 1 \\ c_x & c_y & 1 \end{vmatrix} \right).$$

The value of  $\chi(a, b, c)$  determines whether the three points  $a, b, c$  make a left turn (+1), a right turn (−1), or are aligned (0). When  $\chi(a, b, c) \neq 0$  for all triples  $a, b, c$ , the point set is said to be in *general position* and  $\chi$  is called *uniform*. Among other things, the order type encodes the convex hull of a point set and whether two segments with endpoints in the point set intersect.

*Abstract order types* generalize the notion of order types on planar point sets. Knuth [20] calls uniform abstract order types *CC-systems* and defines them by the following five axioms (we write  $\chi(p, q, r)$  instead of  $\chi(p, q, r) = +1$  and  $\neg\chi(p, q, r)$  instead of  $\chi(p, q, r) = -1$ ):

1. Cyclic symmetry:  $\chi(p, q, r) \implies \chi(q, r, p)$ .
2. Antisymmetry:  $\chi(p, q, r) \implies \neg\chi(p, r, q)$ .
3. Nondegeneracy:  $\chi(p, q, r) \vee \chi(p, r, q)$ .
4. Interiority:  $\chi(t, q, r) \wedge \chi(p, t, r) \wedge \chi(p, q, t) \implies \chi(p, q, r)$ .
5. Transitivity:  $\chi(t, s, p) \wedge \chi(t, s, q) \wedge \chi(t, s, r) \wedge \chi(t, p, q) \wedge \chi(t, q, r) \implies \chi(t, p, r)$ .

Abstract order types are connected to the well-studied mathematical field of oriented matroids. Specifically, if we consider the equivalence class where  $\chi = -\chi$ , then Knuth [20] proves that (equivalence classes of) uniform abstract order types are in one-to-one correspondence with *uniform acyclic rank-3 oriented matroids*. We refer the interested reader to [5] for more information on oriented matroids. An abstract order type  $\chi$  is *realizable* if there exists a point set in  $\mathbb{R}^2$  with order type  $\chi$ . Not all abstract order types are realizable: the smallest non-realizable abstract order type is the well-known Pappus arrangement on 9 points.

Order types are closely related to *pseudoline arrangements*. Pseudoline arrangements are usually considered in the real projective plane  $\mathbb{P}^2$ , where they can be defined as simple closed curves, every pair of which meet in exactly one point [18]. We recall that the projective plane is the extension of the Euclidean plane by a point “at infinity” for each direction  $\alpha$  where the lines with direction  $\alpha$  are defined to intersect, and the line at infinity contains exactly the points at infinity. For an excellent introduction to projective geometry, we refer the interested reader to [28], but we do not assume any familiarity with projective geometry here. Two projective pseudoline arrangements  $A$  and  $A'$  in  $\mathbb{P}^2$  are *isomorphic* if there is a self-homeomorphism of the projective plane that turns  $A$  into  $A'$ .

(Uniform) abstract order types correspond exactly to (simple) projective pseudoline arrangements with a marked face. For straight-line arrangements, the marked face corresponds to the convex hull of the point set described by the order type. For more background on pseudoline arrangements with a marked face and their encodings, the reader is referred to Felsner [12] (Chapter 6).

By the Folkman-Lawrence topological representation theorem [13], equivalence classes of projective pseudoline arrangements correspond in one-to-one fashion to reorientation classes of simple rank-3 oriented matroids [5]. A pseudoline arrangement is *simple* if no three pseudolines meet in the same point. A simple projective pseudoline arrangement is *stretchable* if and only if it is isomorphic to a simple arrangement of straight lines. In 1988, Mnëv proved that

every semialgebraic set is stably equivalent to the realization space of some rank-3 oriented matroid [26]. Furthermore it was shown that the underlying matroid could be made uniform (see also Lemma 4 in Shor [31]). As a by-product of these results, the *simple pseudoline stretchability* problem of deciding stretchability of a simple projective pseudoline arrangement is  $\exists\mathbb{R}$ -complete [29].

We define *uniform order type realizability* as the problem of deciding whether a given uniform abstract order type has a realization. The following lemma summarizes the correspondence between abstract order types and pseudoline arrangements, and the polynomial-time equivalence of the realizability and stretchability problems. The order type realizability problem will be the starting point of our reduction.

**Lemma 1** *Given a uniform abstract order type  $\chi$ , we can compute in polynomial time a description of a simple projective pseudoline arrangement  $A$  with a marked face such that  $\chi$  is realizable if and only if  $A$  is stretchable. Conversely, given a simple projective pseudoline arrangement  $A$  with a marked face, we can compute in polynomial time a uniform abstract order type  $\chi$  such that  $\chi$  is realizable if and only if  $A$  is stretchable.*

### 3 $\exists\mathbb{R}$ -completeness of $k$ -SGE

We first reproduce the reduction from weak rectilinear realizability due to Gassner et al. [14] and then give a direct proof by reduction from the stretchability problem.

#### 3.1 Reduction from weak rectilinear realizability

An *abstract topological graph* (AT-graph) is a pair  $(G, R)$  where  $G = (V, E)$  is a graph and  $R \subseteq \binom{E}{2}$  is a set of pairs of its edges. A straight-line drawing of  $G$  is a *weak rectilinear realization* of  $(G, R)$  if every pair of edges that cross in the drawing is contained in  $R$ . Deciding if an AT-graph has a weak rectilinear realization was shown to be  $\exists\mathbb{R}$ -complete by Kynčl [22].

Kynčl proves  $\exists\mathbb{R}$ -hardness of weak rectilinear realizability by a reduction from simple pseudoline stretchability. Given a simple arrangement  $A$  of  $m$  pseudolines, Kynčl constructs an AT-graph  $(G, R)$  with  $G = (V, E)$  that admits a weak rectilinear realization if and only if  $A$  is stretchable. In this construction, similar to the *order forcing lemma* of Kratochvíl and Matoušek [21], there is one edge associated with each pseudoline, but there is also a pair of edges corresponding to each crossing between two pseudolines.

The weak rectilinear realizability problem is closely related to the  $k$ -SGE problem. The following equivalence is analogous to the equivalence given in Theorem 2 of [14]. Given graphs  $G_1 = (V, E_1), \dots, G_k = (V, E_k)$ , we construct an AT-graph  $(G, R)$  with  $G = (V, \bigcup_i E_i)$  and  $\{e, f\} \in R$  if and only if  $\{e, f\} \not\subseteq E_i$  for all  $i$ . Then  $G_1, \dots, G_k$  admit a  $k$ -SGE if and only if  $(G, R)$  admits a weak rectilinear realization. Conversely, given an AT-graph  $(G, R)$  with  $G = (V, E)$ , we construct a graph  $G_{e,f} = (V, \{e, f\})$  for each pair of edges  $\{e, f\} \notin R$ . Then

$(G, R)$  admits a weak rectilinear realization if and only if the family  $F = \{G_{ef} \mid \{e, f\} \notin R\}$  admits an  $|F|$ -SGE.

Combining Kynčl’s argument with this equivalence yields the following: given an arrangement  $A$  of  $m$  pseudolines, we can construct a set of  $k_m$  graphs  $G_i$ , each on  $n_m$  vertices, such that  $G_1, \dots, G_{k_m}$  admit a  $k_m$ -SGE if and only if  $A$  is stretchable. Here,  $k_m = \Theta(m^4)$  and  $n_m = \Theta(m^2)$ . Fix any constant  $0 < \varepsilon \leq 1$  and add an additional  $m^{4/\varepsilon}$  isolated vertices to each graph  $G_i$ . After this modification,  $n_m = \Theta(m^{4/\varepsilon})$  and thus  $k_m = \Theta(n_m^\varepsilon)$ . Since this takes polynomial time, we obtain the following:

**Theorem 1** *Given graphs  $G_1 = (V, E_1), \dots, G_k = (V, E_k)$  on  $n$  vertices, the decision problem  $k$ -SGE is  $\exists\text{R}$ -complete for  $k = \Omega(n^\varepsilon)$  and any constant  $\varepsilon > 0$ .*

### 3.2 Reduction from uniform order type realizability

We will give an alternative proof of the result from the previous section via a polynomial-time reduction from uniform order type realizability to  $k$ -SGE.

#### 3.2.1 Radial Systems

The high-level idea is to associate one graph with each point  $v$  of the uniform abstract order type  $\chi$  so that a proper geometric embedding of the graph forces some *radial ordering* of the other points around  $v$ . In a set of  $n$  points in the plane, the radial ordering  $R(v)$  associated with the point  $v$  is simply the order in which the  $n - 1$  other points are encountered by a counterclockwise ray sweep around  $v$ . We will define this order up to a circular shift. This is illustrated in Fig. 2(a). We call  $R$  the (*counterclockwise*) *radial system* of  $\chi$ .

The radial system can be inferred from the order type of the point set, and therefore can be defined even if  $\chi$  is not realizable. A way to determine the radial ordering of  $v$  is to pick another point  $w$  and first consider only the points  $x$  such that  $\chi(v, w, x) = +1$ , that is, the points on the left of the oriented line

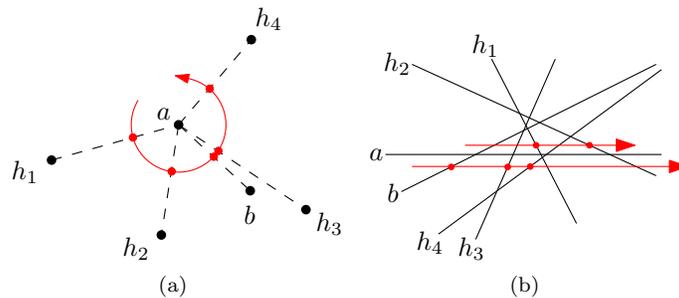


Figure 2: (a) The radial ordering around point  $a$  is  $h_1, h_2, b, h_3, h_4$  (up to a circular shift) [1]. (b) In the dual,  $h_1, h_2$  intersect  $a$  from above in this order and  $b, h_3, h_4$  intersect  $a$  from below in this order.

$vw$ . We can then sort these points using that  $x < x'$  whenever  $\chi(x, v, x') = -1$ . We can similarly sort the points  $x$  such that  $\chi(v, w, x) = +1$  and recover the complete radial ordering  $R(v)$ .

The radial system can also be extracted from the pseudoline arrangement corresponding to the abstract order type. For the pseudoline  $v$ , the corresponding radial ordering is constructed as follows. Starting on any point on the pseudoline  $v$ , we first report the order of the successive intersections with pseudolines coming from the same side as the marked face, followed by the order of the intersections with the pseudolines coming from the other side. In a Euclidean realization of the arrangement, this corresponds to pseudolines coming from *above* and from *below*, respectively. This dual definition of the radial orderings is illustrated on Fig. 2(b). Note that the radial orderings are not the same as the *local sequences* defined by Goodman and Pollack [16]. The local sequence for a pseudoline  $v$  is simply the order of the intersections with the other pseudolines, and correspond in the primal point set to a sweep with a *line* through the point  $v$ , instead of a ray.

The relation between radial systems and order types has been studied in depth in a more general setting in a recent paper from a superset of the current authors [1]. It was shown in particular that the radial orderings alone are *not sufficient* to recover the complete order type of a point set in the plane. Furthermore, for point sets with a triangular convex hull, there can be as many as  $n - 1$  different order types having the exact same radial orderings for each point. The reader is referred to this paper for more results and examples.

It is not too difficult to show, however, that the set of radial orderings is sufficient to recover the order type, provided we also know the points on the convex hull. This is a specialization of Lemma 3 in [1].

**Lemma 2 ([1])** *Consider a realizable abstract order type  $\chi$  on  $n$  points, let  $S$  be the set of counterclockwise radial orderings of the points, and let  $H$  be the set of points on the convex hull. Then the pair  $(S, H)$  uniquely determines  $\chi$ .*

The proof is straightforward and involves three steps. We first recover the order of the points on the convex hull by looking at the radial ordering of one of them. Next, we recover the orientation of every triple with at least one point  $p$  on the convex hull from the radial ordering of  $p$ . Finally, the orientations of the remaining triples are deduced by sweeping a ray around a point on the convex hull and then sweeping a ray around every point encountered.

### 3.2.2 Reduction

Before delving into the construction, we have to argue that we can assume without loss of generality that the convex hull of the input abstract order type  $\chi$  for the realizability problem is triangular. Using Lemma 1, we can compute a projective pseudoline arrangement  $A$  that is stretchable if and only if  $\chi$  is realizable, in such a way that the convex hull of  $\chi$  corresponds to the marked face of  $A$ . If this face is bounded by at most three pseudolines, then we are done. Otherwise, since it is known that every projective arrangement of  $n$  pseudolines

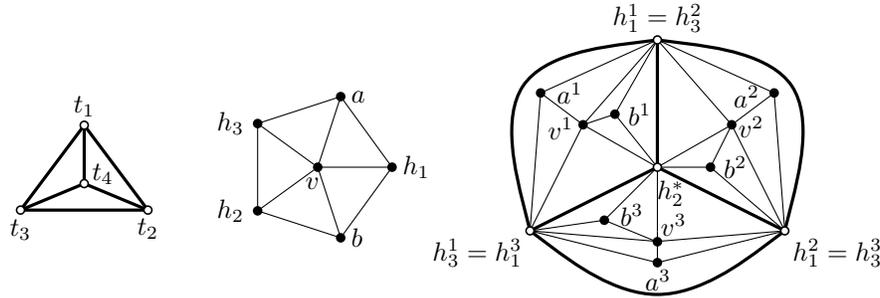


Figure 3: From left to right: the copy of  $K_4$ , a wheel graph  $W_v$  for an order type with convex hull  $h_1, h_2, h_3$ , and the graph  $T_v$ . In  $T_v$  we have  $t_1 = h_1^1 = h_3^2$ ,  $t_2 = h_1^2 = h_3^3$ ,  $t_3 = h_2^1 = h_3^3$ , and  $t_4 = h_2^2 = h_2^3 = h_3^3$ .

has at least  $n$  triangular faces [23], we make such a face the marked face of  $A$ . Applying Lemma 1 in the other direction finally gives us an abstract order type  $\chi'$  with a triangular convex hull that is realizable if and only if  $\chi$  is realizable.

We now have all the ingredients required for the reduction. For each  $v \in V$  we define the *wheel graph*  $W_v$  on  $V$  as the union of the cycle  $R(v)$  corresponding to the radial ordering around  $v$ , and the star connecting  $v$  to all vertices in  $R(v)$ . The purpose of including such a graph is to encode the radial ordering  $R(v)$  of the  $n - 1$  other points around  $v$ .

We next create the labeled graph  $T_v$  by embedding three copies of  $W_v$  into the interior faces of a copy of  $K_4$ , the complete graph on four vertices  $\{t_1, t_2, t_3, t_4\}$ , as shown on the left in Fig. 3. We distinguish the vertices of different copies by adding a superscript  $i$  to the vertices of copy  $i$ . The convex hull  $h_1^1, h_2^1, h_3^1$  is embedded onto  $t_1, t_4, t_3$ ; the convex hull  $h_1^2, h_2^2, h_3^2$  is embedded onto  $t_2, t_4, t_1$ ; and the convex hull  $h_1^3, h_2^3, h_3^3$  is embedded onto  $t_3, t_4, t_2$ . Fig. 3 shows an example of a wheel graph  $W_v$  and the resulting graph  $T_v$ . The graph  $T_v$  has exactly  $3n - 5$  vertices. The reason why we need to embed three copies of the wheel graph  $W_v$ , and not simply one, is that the abstract order type will be preserved only provided the convex hull is the same. We will see that three copies are sufficient to guarantee that at least one of them will have the same convex hull as the one specified by the original abstract order type.

Though the  $T_v$  in the example is maximal planar, this is not always the case. We do, however, have the following.

**Lemma 3** *Each  $T_v$  is 3-connected.*

**Proof:** Using symmetry it is easy to verify that  $W_v$  is 3-connected. We will use Menger’s theorem to prove that  $T_v$  is also 3-connected. Let  $u$  be any vertex of  $W_v$ . From every vertex  $u$  in  $W_v$  there is a path to  $h_1$ , a path to  $h_2$  and a path to  $h_3$  such that the paths share no vertex other than  $u$ . This can be seen as follows. If  $u = v$  then we can reach each  $h_i$  in one step. Otherwise, one path traverses the cycle in a clockwise direction, one traverses the cycle in a counterclockwise direction and one goes via  $v$ . The same holds for the copy

of  $K_4$  (which can also be thought of as a wheel graph). It follows immediately that there are three interior pairwise vertex-disjoint paths between every two vertices in  $T_v$ . Hence, the lemma follows by Menger’s theorem.  $\square$

Since  $T_v$  is 3-connected, all embeddings of  $T_v$  are the same up to reflection and the choice of the outer face. Let  $\mathcal{T}$  be the set of all  $T_v$ .

**Theorem 2** *Given a abstract order type  $\chi$  on a set  $V$  of  $n$  elements, we can compute in polynomial time a set  $\mathcal{T}$  of  $n$  graphs, each on the same set of  $3n - 5$  vertices, such that  $\mathcal{T}$  admits an  $n$ -SGE if and only if  $\chi$  is realizable.*

**Proof:** Suppose that  $\chi$  is realizable. Let  $P$  be a labeled point set that realizes  $\chi$  and let  $p(v)$  be the point in  $P$  that corresponds to  $v$  in  $\chi$ . After possibly reflecting  $P$  along the  $y$ -axis, the counterclockwise ray sweep around each point  $p(v) \in P$  encounters the other points of  $P$  in the order  $R(v)$ . Hence, by construction of the wheel graphs, the induced straight-line drawing of each  $W_v$  on  $P$  is plane. A labeled point set whose induced straight-line drawing of each  $T_v$  is plane can now easily be constructed from three copies of  $P$  and affine transformations.

Conversely, suppose that  $\mathcal{T}$  has an  $n$ -SGE  $\varphi$  and consider its convex hull. Note that the convex hull of  $\varphi$  corresponds to a mutual face of all  $T_v$ : if some  $T_v$  does not have a face that corresponds to the convex hull, then some vertex of  $T_v$  must have been embedded in the outer face of  $T_v$  in  $\varphi$ , which is impossible by Lemma 3. If  $t_1, t_2, t_3$  is the (clockwise) outer face in  $\varphi$ , then the point set corresponding to one of the three copies is a realization of  $\chi$ . This can be seen as follows. If  $t_1, t_2, t_3$  is the outer face in this clockwise order, then the triangle  $h_1^1, h_2^1, h_3^1$  is also oriented in this clockwise order in  $\varphi$ . By Lemma 3, this triangle must form the convex hull of each  $W_v^1$ . Hence, any swap of two elements in any radial ordering  $R(v^1)$  in  $\varphi$  will induce a crossing in the drawing of  $W_v^1$ . It follows that  $\varphi$  is consistent with all radial orderings and therefore, from Lemma 2, it induces a realization of  $\chi$ . If a face other than  $t_1, t_2, t_3$  was chosen to be the outer face in  $\varphi$ , say a face bounded by three vertices of copy one, then the point set corresponding to the vertices of the second copy (or the third; both work) is a realization of  $\chi$  by a similar argument. This concludes the proof.  $\square$

We showed that uniform order type realizability can be reduced in polynomial time to  $k$ -SGE. Since uniform order type realizability is  $\exists\mathbb{R}$ -complete, it follows that  $k$ -SGE is  $\exists\mathbb{R}$ -hard and hence  $\exists\mathbb{R}$ -complete by the fact that  $k$ -SGE belongs to  $\exists\mathbb{R}$  [10]. We finally add a suitable amount of isolated vertices as explained in Subsection 3.1 to complete our alternative proof of Theorem 1.

We define *radial system realizability* as the problem of deciding whether a given system of permutations  $R$  is the radial system of a set of points in  $\mathbb{R}^2$ .

**Observation 1** *Radial system realizability is  $\exists\mathbb{R}$ -complete.*

**Proof:** We prove that radial system realizability is polynomially equivalent to uniform order type realizability.

Consider an abstract order type  $\chi$ . Using the method described at the beginning of this section, compute an abstract order type  $\chi'$  from  $\chi$  in polynomial time such that  $\chi'$  is realizable if and only if  $\chi$  is realizable and  $\chi'$  has a triangular convex hull. Compute the radial system  $R$  of  $\chi'$ . Since  $\chi'$  has a triangular convex hull,  $\chi'$  is the only abstract order type with radial system  $R$  (Theorem 1 in [1]). Hence,  $R$  is realizable if and only if  $\chi$  is realizable.

Conversely, consider a system of permutations  $R$  on a set of  $n$  elements. We compute in polynomial time the set  $T(R)$  of at most  $n - 1$  uniform abstract order types that have  $R$  as their radial system (Theorem 1 and Corollary 1 in [1]). Then  $R$  is realizable if and only if at least one uniform abstract order type in  $T(R)$  is realizable.  $\square$

## 4 Simultaneous geometric embeddings requiring doubly exponential grids

Many graph drawing questions involve drawing graphs on a small grid. Our construction gives insight on the following simple problem: given a collection of  $k$  graphs on  $n$  vertices which admit a  $k$ -SGE, can we provide any guarantee on the size of the largest grid on which we can embed them?

Since the decision problem is  $\exists\mathbb{R}$ -hard, it is unlikely that simultaneous geometric embeddings can all be drawn on a small grid. In fact, showing that all collections of  $k = \text{poly}(n)$  graphs that admit a  $k$ -SGE, admit a  $k$ -SGE on a grid of size at most exponential in a polynomial in  $n$  would directly imply that  $\exists\mathbb{R} = NP$ , since the drawing could be encoded using a polynomial number of bits and used as a certificate.

Our construction implies the following lower bound on the size of the smallest grid required for a  $k$ -SGE.

**Theorem 3** *There exist collections of  $k$  graphs on  $n = \Theta(k)$  vertices that admit a  $k$ -SGE, every  $k$ -SGE of which requires a grid of size  $2^{2^{\Omega(n)}}$ .*

**Proof:** We use a well-known construction due to Goodman, Pollack, and Sturmfels [17]. They construct a set of  $k$  points in the plane such that every realization of its order type requires a grid of size  $2^{2^{\Omega(k)}}$ . This construction implements an iterative squaring procedure using the multiplication gadget from Von Staudt's algebra of throws [27, 25].

Let  $\chi$  be an order type on such a set of  $k$  points requiring a doubly exponential-size grid. By Theorem 2, we can construct graphs  $T_1, \dots, T_k$  where each  $T_i$  has  $n = 3k - 5$  vertices, such that every point set  $P$  that admits an  $k$ -SGE of  $T_1, \dots, T_k$ , where  $|P| = n$ , will contain a copy of a realization of  $\chi$ . By definition of  $\chi$ ,  $P$  cannot be represented with points of integer coordinates smaller than  $2^{2^{\Omega(n)}}$ .  $\square$

In the same paper, Goodman, Pollack, and Sturmfels [17] prove that every realizable order type in general position has a realization with coordinates bounded

by  $2^{2^{O(n)}}$ . If a set of graphs admits a  $k$ -SGE, then the resulting point set can be perturbed into general position without introducing any crossing. Finally, since the order type of a point set determines whether two segments cross, it follows that a  $k$ -SGE never requires coordinates larger than  $2^{2^{O(n)}}$ . Hence, Theorem 3 is tight.

This is a significant improvement compared to what can be extracted from the construction of Kynčl [22]. In the latter, an arrangement of  $m$  pseudolines is realized via the simultaneous geometric embedding of graphs on a set of  $k_m$  graphs on  $n_m = \Theta(m^2)$  vertices. Hence although the coordinates may have value  $2^{2^{\Omega(m)}}$ , this is only  $2^{2^{\Omega(\sqrt{n_m})}}$ .

## 5 Concluding remarks

We gave an alternative proof for the  $\exists\mathbb{R}$ -hardness of  $k$ -SGE and we showed that a  $k$ -SGE may sometimes need a grid of size  $2^{2^{\Omega(n)}}$ . Our hardness proof relies on choosing  $k = \Omega(n^\varepsilon)$ , and it is not clear how to weaken this requirement. The complexity of the cases  $k = O(\log n)$  and in particular  $k = 2$  are still open, including whether these problems are in NP.

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