

Equitable colorings of K_4 -minor-free graphs

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Abstract

We demonstrate that for every positive integer Δ , every K_4 -minor-free graph with maximum degree Δ admits an equitable coloring with k colors where $k \geq \frac{\Delta+3}{2}$. This bound is tight and confirms a conjecture by Zhang and Wu. We do not use the discharging method but rather exploit decomposition trees of K_4 -minor-free graphs.

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1 Introduction

Equitable coloring is an ubiquitous notion. From a combinatorial point of view, it corresponds to a natural variation of usual graph coloring where the color classes are required to all have the same size, plus/minus one vertex. Practically, this is one way to prevent color classes from being very large, which can be useful when using graph coloring for scheduling purposes for instance. Theoretically, equitable colorings were used successfully in *a priori* unrelated topics, such as probability. Indeed, one of the seminal results regarding equitable colorings is the following theorem, which was established by Hajnal and Szemerédi [3] (the statement was first conjectured by Erdős).

Theorem 1 (Hajnal–Szemerédi, 1970) *Every graph with maximum degree at most Δ admits an equitable coloring using $\Delta + 1$ colors.*

Theorem 1 allowed for a simplified demonstration of the Blow-up lemma — found by Rödl and Ruciński [11]. In addition, this theorem was also used to derive deviation bounds for sums of random variables with some degree of dependence — this was done by Alon and Füredi [1] and by Janson and Ruciński [4]. Let us point out that in 2010, that is, forty years after Theorem 1 was proved, a much simpler demonstration was finally found, building on several other related results. More precisely, Kierstead, Kostochka, Mydlarz and Szemerédi [6] managed to find a two-page proof of Theorem 1, which also has the advantage to lead to a polynomial-time algorithm that efficiently finds a relevant coloring — contrary to the original argument (see also the independent article by Kierstead and Kostochka [5]).

As it turns out, the notion of equitable colorings behaves pretty differently from usual colorings, and it is a challenging task to better comprehend its relation to well-known graph classes. Starting from graphs with bounded maximum degree, it is natural to consider next d -degenerate graphs. The following theorem was established by Kostochka and Nakprasit [8], in a more general form.

Theorem 2 (Kostochka–Nakprasit, 2003) *Fix an integer Δ greater than or equal to 54. If G is a 2-degenerate graph with maximum degree at most Δ , then G is equitably k -colorable whenever $k \geq \frac{\Delta+3}{2}$.*

Theorem 2 partially confirms a conjecture¹ by Zhang and Wu [12, Conjecture 9], (also see [10, Conjecture 6, p. 1209]) that if $\Delta \geq 3$, then every series-parallel graph with maximum degree Δ admits an equitable k -coloring whenever $k \geq \frac{\Delta+3}{2}$. Indeed, series-parallel graphs are known to be 2-degenerate, so Theorem 2 yields that the conjecture is true if $\Delta \geq 54$. The purpose of our work is to establish the conjecture for all the remaining cases, that is, $\Delta \in \{3, \dots, 53\}$. (Although, in our proofs we do not use the upper bound on Δ , and simply prove the statement for all K_4 -minor-free graphs.)

The statement conjectured by Zhang and Wu is actually a strengthening of a result of theirs [12], which establishes that every series-parallel graph with

¹There is a small misprint in the printed versions of the conjecture: the printed bound is $\frac{\Delta}{2}$, which is trivially false as we shall see later; the authors definitely meant $\frac{\Delta+3}{2}$.

maximum degree $\Delta \geq 3$ admits an equitable k -coloring if $k \geq \Delta$. The conjecture can also be seen as a generalisation of a theorem of Kostochka [7] that every outerplanar with maximum degree $\Delta \geq 3$ admits an equitable k -coloring whenever $k \geq \frac{\Delta+3}{2}$.

It is worth mentioning that Kostochka, Nakprasit and Pemmaraju [9] established (a generalisation of) the following interesting statement.

Theorem 3 (Kostochka, Nakprasit & Pemmaraju, 2005) *If k is an integer greater than 123 and G is a 2-degenerate graph with maximum degree at most $\frac{1}{2}|V(G)| + 1$, then G admits an equitable k -coloring.*

Theorem 3, however, does not bring us any new information regarding the problem at hands. Indeed, we need to consider graphs with maximum degree $\Delta \leq 53$, while the number of colors needs to be at least 124. Hence for our question the information provided by Theorem 3 is already contained in the aforementioned result of Zhang and Wu [12].

As reported earlier we establish the following, which confirms Zhang and Wu’s conjecture because a series-parallel graph has no K_4 -minor [2].

Theorem 4 *If G is a K_4 -minor-free graph with maximum degree Δ , then G admits an equitable k -coloring whenever $k \geq \frac{\Delta+3}{2}$.*

We point out that the lower bound given in Theorem 4 is tight, as shown by the following example. Fix a positive integer Δ and let $C(\Delta - 1)$ be obtained from the complete graph K_2 on two vertices u and v by adding $\Delta - 1$ paths of length two between u and v (see Figure 1 for a representation of $C(6)$). The graph $C(\Delta - 1)$ is series-parallel, and hence K_4 -minor-free. In any equitable coloring c of $C(\Delta - 1)$, the colors $c(u)$ and $c(v)$ are different and used on no other vertices of G . Therefore, the number of colors used by c is at least $\lceil \frac{\Delta+3}{2} \rceil$.

Contrary to the proof of some of the results mentioned above, we do not rely on discharging, but rather on the structural links between K_4 -minor-free graphs and two-terminal series-parallel graphs: in particular, our proof heavily relies on a so-called SP-tree. Before proceeding with the proof, we review some folklore properties of K_4 -minor-free graphs and two-terminal series-parallel graphs and introduce a bit of terminology.

It would be interesting to know whether Theorem 4 can be extended to the class of 2-degenerate graphs. A generalisation of this has actually been conjectured in 2003 by Kostochka and Napkrasit [8].

Conjecture 5 *Fix an integer Δ . If $d \in \{2, \dots, \Delta\}$ and G is a d -degenerate graph with maximum degree at most Δ , then G admits an equitable k -coloring whenever $k \geq \frac{\Delta+d+1}{2}$.*

2 The structure of K_4 -minor-free Graphs

As it turns out, graphs with no K_4 -minor are strongly related to two-terminal series-parallel graphs. A *two-terminal graph* is a graph with two distinguished

vertices called *poles*. We consider two basic operations on such graphs: the serial join and the parallel join. For $i \in \{1, 2\}$, let G_i be a two-terminal graph with poles u_i and v_i . The graph $S(G_1, G_2)$ obtained by identifying the vertices v_1 and u_2 is also a two-terminal graph and its two poles are the vertices u_1 and v_2 . The graph $S(G_1, G_2)$ obtained in this way is called the *serial join* of G_1 and G_2 . The *parallel join* of G_1 and G_2 is the graph $P(G_1, G_2)$ obtained by identifying the pairs of vertices (u_1, u_2) and (v_1, v_2) ; the poles of $P(G_1, G_2)$ being the identified vertices. If G_1, \dots, G_m are two-terminal graphs where $m \geq 3$, we inductively define $P(G_1, \dots, G_m) := P(P(G_1, \dots, G_{m-1}), G_m)$. In other words, $P(G_1, P(G_2, \dots, P(G_{m-1}, G_m) \dots))$. *Two-terminal series-parallel graphs* are two-terminal graphs that can be obtained by the following recursive construction². The basic two-terminal series-parallel graph is an edge uv with the two poles being its end-vertices. Two-terminal series-parallel graphs are precisely those that can be obtained from edges by a sequence of serial and parallel joins. A two-terminal series-parallel graph usually admits several different constructions.

It is also well known that every 2-edge-connected K_4 -minor-free graph is a two-terminal series-parallel graph [2, Theorem 2]. Consequently, the set of K_4 -minor-free graphs can also be seen as the closure of two-terminal series-parallel graphs by the spanning subgraph relation.

Lemma 1 *A graph G has no K_4 -minor if and only if G is the spanning subgraph of a two-terminal series-parallel graphs.*

To see Lemma 1, note that spanning subgraphs of two-terminal series-parallel graphs have no K_4 -minor. For the reverse direction, it suffices to notice that a K_4 -minor-free graph in which adding any new edge creates a K_4 -minor is 2-edge-connected.

As a consequence, K_4 -minor-free graphs are precisely those for which we can choose two poles such that the two-terminal graph obtained can be constructed from the complete graph K_2 on two vertices and its complement $\overline{K_2}$. The construction of a particular K_4 -minor-free graph G can thus be encoded by a rooted tree, which is called the *SP-decomposition tree* of G . Each node of the tree corresponds to a subgraph of G obtained at a step of the recursive construction of G . The leaves correspond to graphs with only two poles (and no other vertex) that may or may not be connected by an edge. Each inner node of the tree corresponds to either a serial join or to a parallel join. Based on this, there are two types of inner nodes: *S-nodes* and *P-nodes*. The inner nodes have at least two children: the subgraphs corresponding to their children are joined together by a sequence of serial or parallel joins depending on the type of the node. Since the result of a sequence of serial joins depends on the order in which the serial joins are applied, the children of each inner node are ordered. Without loss of generality, we can assume that the children of a P-node are S-nodes and leaves only, and the children of an S-node are P-nodes and leaves only.

²We point out that in the literature, such graphs are sometimes called simply 'series-parallel graphs', while this term can also be used to refer to K_4 -minor-free graphs.

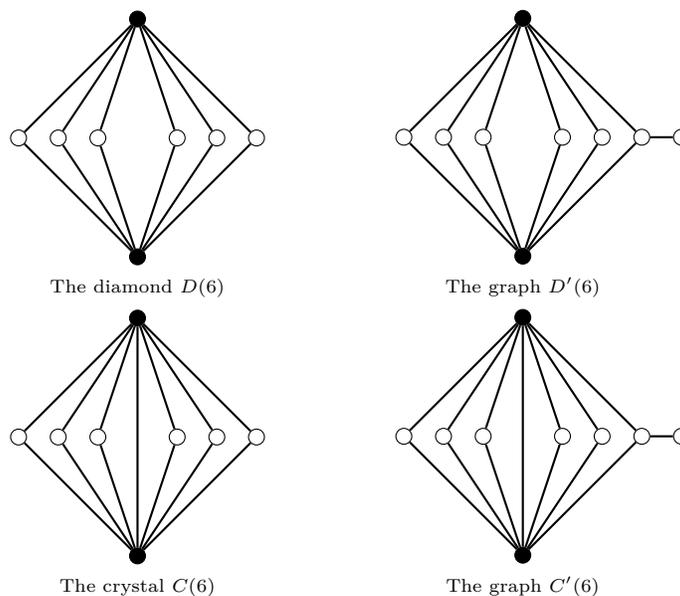


Figure 1: Illustrations of specific two-terminal K_4 -minor-free graphs, the filled vertices being the two poles of each graph.

If A is a two-terminal graph, the vertices of A distinct from its poles are said to be its *inner* vertices. The set of inner vertices of A is $\text{Inner}(A)$. We define $\text{width}(A)$, the *width* of A , to be the number of inner vertices of A , that is, $\text{width}(A) = |\text{Inner}(A)|$ (note that $\text{width}(A) = |V(A)| - 2$). We introduce some terminology for particular two-terminal K_4 -minor-free graphs (see Figure 1 for illustrations). A two-terminal graph obtained by a parallel join of several two-edge paths is a *diamond*. A two-terminal graph obtained by a parallel join of several two-edge paths and an edge is a *crystal*. Observe that an edge may be seen as a crystal of width 0. If i is a positive integer, we define $D(i)$ to be the diamond with width i and $C(i)$ to be the crystal with width i . Let $D'(1)$ be the graph $K_{1,3}$ with two vertices of degree 1 as poles. For $i \geq 2$, we define $D'(i)$ to be the graph obtained by a parallel join of $D'(1)$ with $i - 1$ paths of length 2. Let $C'(i)$ be obtained from $D'(i)$ by adding an edge between the poles. We let P_i be the path with i vertices. If G is a graph and U a subset of the vertices of G , we let $G - U$ be the subgraph of G induced by the vertices of G that do not belong to U . For a positive integer k , we take the representatives of \mathbf{Z}_k to be $\{1, \dots, k\}$, rather than the more common $\{0, \dots, k - 1\}$. An *equitable k -coloring* of a graph G is a mapping $\alpha: V(G) \rightarrow \mathbf{Z}_k$ such that $|\alpha^{-1}(\{i\})|$ and $|\alpha^{-1}(\{j\})|$ differ by at most one for every $(i, j) \in \mathbf{Z}_k^2$.

The next lemma is a simple but useful remark about common neighbors of the poles of a K_4 -minor-free graph.

Lemma 2 *If H is a K_4 -minor-free two-terminal graph with poles a and b given by an SP-decomposition tree, then every two distinct vertices in $N_H(a) \cap N_H(b)$ belong to different components of $H \setminus \{a, b\}$. In particular, $N_H(a) \cap N_H(b)$ is an independent set of H .*

Proof: Using induction on the number of vertices of the SP-decomposition tree of H , we prove that no two vertices in $N_H(a) \cap N_H(b)$ belong to a same component of $H \setminus \{a, b\}$.

- The statement is trivial if the SP-tree has only one node, that is if H has two vertices.
- If H is the series join of H_1 and H_2 , then the only possible common neighbor of a and b is the common pole of H_1 and H_2 . The statement is therefore true in this case also.
- If H is the parallel join of H_1 and H_2 , then let x and y be two common neighbors of a and b . Either x and y belong to H_i for some $i \in \{1, 2\}$, in which case the result follows from the induction hypothesis applied on H_i ; or x and y are in different components of $H \setminus \{a, b\}$.

□

Let T be an SP-decomposition tree (of a K_4 -minor-free graph), and n be a node of T representing the subgraph H with poles a and b . Assume that $H - \{a, b\}$ has m components C^1, \dots, C^m . The node n is in *normal form* if $m \leq 1$ (i.e. $H - \{a, b\}$ either is connected or has no vertex at all), or if n is a parallel node with children H^1, \dots, H^m plus the edge ab if $ab \in E(H)$, where H^i is the subgraph of H induced by $C^i \cup \{a, b\}$ from which we remove the edge ab if it is present. The tree T is in *normal form* if every node of T is in normal form.

Lemma 3 *If G is a K_4 -minor-free graph, then G admits a construction tree in normal form.*

Proof: As a K_4 -minor-free graph, G has two vertices a and b and an SP-decomposition tree T that represents the two-terminal graph G with poles a and b . Note that we may assume that T is a binary tree (where P-nodes and S-nodes may not alternate).

To prove the lemma, we describe an inductive procedure that transforms the (binary) SP-decomposition tree T into an SP-decomposition tree T' in normal form that represents the same graph G . Assume that this procedure exists for trees with fewer nodes than T . If n is a leaf, then G has two vertices and further $V(G) - \{a, b\}$ is empty, so n is in normal form indeed. So we now suppose that n has two children representing the graphs G_1 and G_2 , respectively. By induction, for each $i \in \{1, 2\}$ there is a tree T_i in normal form that represents G_i . We distinguish two cases depending on the type of the root n of T .

- Suppose that n is a P-node, so $G = P(G_1, G_2)$. Let $C_i^1, \dots, C_i^{m_i}$ be the components of $G_i - \{a, b\}$, and note that m_i is a positive integer. If $m_i = 1$, then we set $H_i^1 := G_i$. If $m_i \geq 2$, then according to the definition of normal forms the graph G_i is encoded in T_i by the parallel join of $H_i^1, \dots, H_i^{m_i}$, plus possibly the edge ab . (We recall that it means that each graph H_i^j is the subgraph of G_i induced by $C_i^j \cup \{a, b\}$ from which the edge ab is deleted if it is present.) The sought SP-decomposition tree T' is then obtained by making a new P-node n the parent node of each of the SP-decomposition trees representing $H_1^1, \dots, H_1^{m_1}, H_2^1, \dots, H_2^{m_2}$ (each of them in normal form), and, possibly, of a leaf representing an edge if $ab \in E(G)$.
- Suppose that n is an S-node, so $G = S(G_1, G_2)$. First note that $ab \notin E(G)$. Let c be the common pole of G_1 and G_2 . Let $C_1^1, \dots, C_1^{k_1}$ be the components of $G_1 - \{a, c\}$ that contain a neighbor of c and let $C_1^{k_1+1}, \dots, C_1^{m_1}$ be the other components of $G_1 - \{a, c\}$. We define analogously the components $C_2^1, \dots, C_2^{m_2}$ and the index k_2 with respect to $G_2 - \{b, c\}$. For each $j \in \{1, \dots, m_1\}$, we define H_1^j to be the subgraph of G corresponding to the component C_1^j of $G_1 - \{a, c\}$ as in the definition of normal forms. The graphs $H_2^1, \dots, H_2^{m_2}$ are defined analogously with respect to $G_2 - \{b, c\}$.

According to the definition of normal forms, either $H_i^1 = G_i$ or, in T_i , the graph G_i is represented by $P(H_i^1, \dots, H_i^{m_i})$. Note that the components of $G - \{a, b\}$ are exactly $C_1^{k_1+1}, \dots, C_1^{m_1}, C_2^{k_2+1}, \dots, C_2^{m_2}$ and $\{c\} \cup (\bigcup_{j=1}^{k_1} C_1^j) \cup (\bigcup_{j=1}^{k_2} C_2^j)$. Based on this, the sought tree T' is easily, albeit tediously, obtained. For $i \in \{1, 2\}$, we set

$$F_i := \begin{cases} P(H_i^1, \dots, H_i^{k_i}) & \text{if } k_i \geq 2, \\ H_i^1 & \text{if } k_i = 1, \\ \overline{K_2} & \text{if } k_i = 0. \end{cases}$$

If $k_1 = m_1$ and $k_2 = m_2$, then $G = S(F_1, F_2)$ and hence the sought tree T' is obtained from an SP-decomposition tree T_1 of F_1 and an SP-decomposition tree T_2 of F_2 , each in normal form, by adding an S-node with children T_1 and T_2 . The tree T' is in normal form because $G - \{a, b\}$ is connected and T_1 and T_2 are in normal form.

Otherwise, the sought tree T' is the tree with a P-node as a root, whose children are the SP-decomposition trees representing $H_1^{k_1+1}, \dots, H_1^{m_1}, H_2^{k_2+1}, \dots, H_2^{m_2}$ and $S(F_1, F_2)$, each of them in normal form. It follows from the construction that the node n is in normal form, hence so is the tree T' . This concludes the proof.

□

3 Reductions

We note that the statement of Theorem 4 is true if $k \leq 2$, since then $\Delta \in \{0, 1\}$. So from now on we assume that $k \geq 3$. We fix a minimal counter-example (G, k) , where $k \geq \lceil \frac{\Delta(G)+3}{2} \rceil$, along with an SP tree-decomposition T of G with every node in normal form (Lemma 3 ensures that this is possible). It follows that $k < |V(G)|$, as any graph H admits an equitable t -coloring if $t \geq |V(H)|$. If G is the disjoint union of two graphs G_1 and G_2 each of which admit an equitable k -coloring, then one can deduce an equitable k -coloring of G as follows. There is an equitable k -coloring of G_1 where each of the colors $1, \dots, m_1$ is used once more than each color of $m_1 + 1, \dots, k$ for some integer $m_1 \in \{1, \dots, k + 1\}$. Similarly, there is an equitable k -coloring of G_2 such that the colors $1, \dots, m_2$ are all used once less than the colors $m_2 + 1, \dots, k$ for some integer $m_2 \in \{0, \dots, k\}$. The union of these two colorings is an equitable k -coloring of G . As a consequence, we may assume that G is connected. It also follows that every component of a subgraph of G with poles a and b that is represented by a subtree of T contains a or b . A subtree T' of T is a *construction subtree* if T' is rooted at a node r of T and $T' - \{r\}$ consists of at least two subtrees of $T - \{r\}$ containing children of r such that if r is an S-node, then all these children are consecutive around r in T . To ease the reading, let us summarize our assumptions and deductions on G and k :

1. $3 \leq k < |V(G)|$;
2. G is connected; and
3. every component of a subgraph of G with poles a and b that is represented by a subtree of T contains a or b .

Throughout this section, each time a coloring c is obtained by induction (or, equivalently, by a minimality argument), we assume the colors to be ordered increasingly, that is, such that $|\alpha^{-1}(\{i\})| \leq |\alpha^{-1}(\{j\})|$ for every two colors i and j with $i < j$. (This condition implies that if we consider a k -coloring α of an n' -vertex graph with $n' < k$, then the colors used by c are precisely $k, k - 1, \dots, k - n'$, each being used exactly once.)

Lemma 4 *The graph G has no construction subtree representing a subgraph isomorphic to $C(k - 1)$ or to $D(k - 1)$.*

Proof: Suppose, on the contrary, that H is such a subgraph of G . Let a and b be the poles and v_1, \dots, v_{k-1} the inner vertices of H . Let F be the graph constructed from G by contracting $V(H)$ to a vertex c , removing parallel edges and loops when they occur. Note that F has no K_4 -minor. In addition, $d_F(c) \leq d_G(a) + d_G(b) - 2(k - 1) \leq 2k - 4$. By the minimality of G , there is an equitable k -coloring α of F . Define $\alpha'(v) := \alpha(v)$ for $v \in V \setminus V(H)$. Note that α' is a partial proper coloring of G , that is, a proper coloring defined on a subset of $V(G)$. To finish the proof, it suffices to extend α' to a proper coloring of G such that the multisets $\{\alpha'(a), \alpha'(b), \alpha'(v_1), \dots, \alpha'(v_{k-1})\}$ and $\{\alpha(c), 1, \dots, k\}$

are equal. (Note that in this latter multiset one color has multiplicity two — namely $\alpha(c)$ — and $k - 1$ colors have multiplicity one.) We now distinguish two cases.

- If $ab \notin E$, then we set $\alpha'(a) := \alpha'(b) := \alpha(c)$ and we color v_1, \dots, v_{k-1} using all the elements of the set $\{1, \dots, k\} \setminus \{\alpha(c)\}$.
- If $ab \in E$, then a has at most $\Delta - k \leq k - 3$ colored neighbors. So a can be properly colored with a color $\alpha'(a)$ different from $\alpha(c)$. Similarly, b has at most $k - 2$ colored neighbors (including a), so b can be properly colored with a color $\alpha'(b)$ different from $\alpha(c)$ (and from $\alpha'(a)$). Now, we color v_1, \dots, v_{k-1} using the elements of the multiset $\{\alpha(c), 1, \dots, k\} \setminus \{\alpha'(a), \alpha'(b)\}$, with the corresponding multiplicities.

□

Corollary 6 *For every integer $t \geq k - 1$, the graph G has no construction subtree representing a subgraph $C(t)$ or $D(t)$.*

Proof: Assume otherwise that H is such a subgraph of G . Let a and b be the poles of H . Let n be the root of the construction subtree that represents H . Since n is in normal form and $H - \{a, b\}$ is an independent set of size t , the node n is a parallel node with at least t children representing a path P_3 with end vertices a and b (the node n may have other children as well). Choosing n as a root along with $k - 1$ of the children of n representing a P_3 yields a construction subtree of T that represents $D(k - 1)$, which contradicts Lemma 4. □

Lemma 5 *If a construction subtree of T represents a graph H such that $1 \leq \text{width}(H) \leq k$, then $\text{Inner}(H)$ is dominated by a pole of H unless $\text{width}(H) \geq 2$ and $H \in \{C'(t), D'(t)\}$, where $t = \text{width}(H) - 1$.*

Proof: Assume that each of the poles a and b of H has a non-neighbor in $\text{Inner}(H)$, which we name a' and b' , respectively. Note that it is possible to ensure that $a' \neq b'$ unless $\text{Inner}(H) \setminus N(a) = \text{Inner}(H) \setminus N(b) = \{a'\}$. In this latter case, since each component of H contains a or b by (3), we deduce that H is connected. Since H has no K_4 -minor, it then follows from Lemma 2 that H is isomorphic to either $C'(t)$ or $D'(t)$, with $t = \text{width}(H) - 1 \geq 1$.

We now assume that $a' \neq b'$, which yields to a contradiction. Indeed, let F be the graph $G - \text{Inner}(H)$ to which we add the edge ab if it is not already present. Note that the addition of the edge ab cannot create a K_4 -minor, because G is K_4 -minor-free and contains a path of length two between a and b that does not belong to F . By the minimality of G there is an equitable k -coloring α of F . To obtain a contradiction, it suffices to extend α to a proper coloring of G such that $\{\alpha(v) \mid v \in \text{Inner}(H)\}$ equals $\{1, \dots, \text{width}(H)\}$. (We recall that the colors are increasingly ordered.)

To do so, we define $\alpha(a') := \alpha(a)$ if $\alpha(a) \leq \text{width}(H)$ and $\alpha(b') := \alpha(b)$ if $\alpha(b) \leq \text{width}(H)$ and we arbitrarily assign the colors of $\{1, \dots, \text{width}(H)\} \setminus \{\alpha(a), \alpha(b)\}$ to the non-colored vertices, each color being assigned once. □

Our next statement is a direct consequence of Lemma 5.

Corollary 7 *If a construction subtree of T represents a graph H of width at most k , then the subgraph induced by $\text{Inner}(H)$ is a forest.*

Proof: The statement is clear if $H \in \{C'(t), D'(t)\}$ for some integer t , so by Lemma 5 we can assume that $\text{Inner}(H)$ is dominated by a pole a of H . Then $\text{Inner}(H)$ induces an acyclic graph, as otherwise $\text{Inner}(H) \cup \{a\}$ would induce a subgraph of G containing a subdivision of K_4 . \square

Lemma 6 *Let H be a graph with poles a and b represented by a construction subtree of T and assume that $\text{width}(H) = k - 1$. Then $d_H(a) + d_H(b) \leq 2k - 4$.*

Proof: Assume on the contrary that $d_H(a) + d_H(b) \geq 2k - 3$. Let F be the graph obtained from G by contracting H into one vertex c , again removing parallel edges and loops when they occur. In other words, we set $V(F) := (V(G) \setminus V(H)) \cup \{c\}$ and $N_F(v) := N_G(v)$ for $v \in V(G) \setminus V(H)$ while $N_F(c) := (N_G(a) \cup N_G(b)) \cap V(F)$. By our assumption, $d_G(c) \leq d_G(a) - d_H(a) + d_G(b) - d_H(b) \leq 2\Delta - (2k - 3) \leq \Delta$. Consequently, F is a K_4 -minor-free graph with maximum degree at most Δ . By the minimality of G there is an equitable k -coloring α of F . To obtain an equitable colouring of G , it suffices to extend α to $V(G)$ in such a way that the multisets $\{\alpha(v) \mid v \in V(H)\}$ and $\{\alpha(c), 1, \dots, k\}$ are equal. We note that Corollary 7 yields that $\text{Inner}(H)$ induces an acyclic graph. We distinguish three cases.

- If $ab \notin E(G)$ then we define $\alpha(a) := \alpha(b) := \alpha(c)$ and we arbitrarily distribute all the colors in $\{1, \dots, k\} \setminus \{\alpha(c)\}$ to the vertices in $\text{Inner}(H)$.
- If $ab \in E(G)$ and a has a non-neighbor $a' \in \text{Inner}(H)$, then by Lemma 5, it follows that either b dominates $\text{Inner}(H)$ or $H = C'(k - 2)$. In both cases, we know that b has at least $k - 2$ neighbors in $\text{Inner}(H)$. It follows that b has at most $\Delta - (k - 2) \leq k - 1$ neighbors outside of $\text{Inner}(H)$, including a . We define $\alpha(a) := \alpha(a') := \alpha(c)$. By the preceding remark it is possible to properly color b with a color $\alpha(b)$ (so in particular $\alpha(a) \neq \alpha(b)$). To finish the coloring, we assign arbitrarily all the colors in $\{1, \dots, k\} \setminus \{\alpha(a), \alpha(b)\}$ to the vertices in $\text{Inner}(H) \setminus \{a'\}$.
- If both a and b dominate $\text{Inner}(H)$, then by Lemma 2 we know that $H = C(k - 1)$, which does not occur by Lemma 4.

\square

Lemma 7 *If H is a graph represented by a construction subtree of G , then $\text{width}(H) \neq k - 1$.*

Proof: Assume otherwise that there is such a graph H with width $k - 1$. By Lemma 5, we may assume that a pole a of H has at least $k - 2$ neighbors in $\text{Inner}(H)$. Let b be the other pole of H . By Lemma 6, we have $d_H(b) \leq 2k - 4 - d_H(a) \leq k - 2$. It follows that b has a non-neighbor b' in $\text{Inner}(H)$.

By the minimality of G , the graph $F := G - (\text{Inner}(H) \cup \{a\})$ has an equitable k -coloring α . To finish the proof, it suffices to extend α to $V(G)$ in such a way that $\{\alpha(v) \mid v \in \text{Inner}(H) \cup \{a\}\}$ equals $\{1, \dots, k\}$. Since a has at most $k - 1$ colored neighbors, it is possible to properly color a with a color $\alpha(a)$. We set $\alpha(b') := \alpha(b)$ unless $\alpha(a) = \alpha(b)$. Then we arbitrarily color the $(k - 1$ or $k - 2)$ non-colored vertices using all the $(k - 1$ or $k - 2)$ colors in $\{1, \dots, k\} \setminus \{\alpha(a), \alpha(b)\}$. \square

Corollary 8 *If H is a graph represented by a construction subtree of G , then $H \notin \{C'(k - 1), D'(k - 1)\}$.*

Proof: Assume otherwise that H is such a graph, with poles a and b , and represented by a construction subtree of G with root n . Since n is in normal form and $H - \{a, b\}$ is disconnected, the node n is a parallel node with a children representing a star $K_{1,3}$ and (at least) $k - 2$ children each representing a path P_3 with end-vertices a and b (the node n may have further children). It follows that T has a construction subtree of G rooted on n representing $D'(k - 2)$, which has width $k - 1$. This contradicts Lemma 7. \square

Lemma 8 *If H is a graph represented by a construction subtree of G , then $\text{width}(H) \neq k$.*

Proof: Suppose, on the contrary, that H is such a graph with width k . Let a and b be the poles of H . By Lemmas 6 and 8, we know that $H \notin \{C(k), C'(k - 1), D(k), D'(k - 1)\}$. It now follows from Lemma 5, that a dominates $\text{Inner}(H)$. Then b has a non-neighbor $b' \in \text{Inner}(H)$, for otherwise b also would dominate $\text{Inner}(H)$, so Lemma 2 would imply that $H \in \{C(k), D(k)\}$.

Let F be the graph $G - \text{Inner}(H)$ to which we add the edge ab if it is not already present. By the minimality of F there is an equitable k -coloring α of F . To finish the proof, it suffices to deduce a proper coloring α' of G that equals α on $V(G) \setminus (\text{Inner}(H) \cup \{a\})$ and such that the multisets $\{1, \dots, k\} \cup \{\alpha(a)\}$ and $\{\alpha'(u) \mid u \in \text{Inner}(H) \cup \{a\}\}$ are equal. We distinguish two cases depending on the value of k .

- Case 1: $k \geq 4$. Since a has k neighbors in $\text{Inner}(H)$, the vertex a has at most $\Delta - k \leq k - 3$ colored neighbors, so we can properly recolor a with a color $\alpha'(a)$ different from both $\alpha(a)$ and $\alpha(b)$. By Corollary 7, $\text{Inner}(H)$ is a forest and we know that $|\text{Inner}(H) \setminus \{b'\}| = k - 1 \geq 3$, so there is an independent set $A \subset \text{Inner}(H) \setminus \{b'\}$ of size 2. To complete the coloring, we assign $\alpha(b)$ to b' and $\alpha(a)$ to the vertices in A and we distribute arbitrarily the colors in $\{1, \dots, k\} \setminus \{\alpha'(a), \alpha(a), \alpha(b)\}$ to the non-colored vertices.
- Case 2: $k = 3$. Since a dominates a set of size k , it holds that $k \leq \Delta \leq 2k - 3$, so $k = 3 = \Delta$. Moreover, it also follows that $ab \notin E$. As a consequence of Corollary 7, the set $\text{Inner}(H)$ contains two non-adjacent vertices v_1 and v_2 . Let u be the third vertex in $\text{Inner}(H)$, so $\text{Inner}(H) = \{v_1, v_2, u\}$. We define $\alpha'(a) := \alpha(b)$, we set $\alpha'(v_i) := \alpha(a)$ for $i \in \{1, 2\}$ and we attribute to u the third color, that is the one in $\{1, 2, 3\} \setminus \{\alpha(a), \alpha(b)\}$.

In both cases, we obtain an equitable k -coloring of G , a contradiction. \square

Our last two lemmas rely on the following observation.

Observation 9 *Let m be a positive integer and let $\lambda_1, \dots, \lambda_m \in \{1, 2\}$. If A_1 and A_2 are two subsets of the vertices of a graph G that has no edge between A_1 and A_2 , then the vertices in $A_1 \cup A_2$ can be properly colored using the colors $1, \dots, m$ with respective multiplicities $\lambda_1, \dots, \lambda_m$ whenever $\sum_{j=1}^m \lambda_j = |A_1| + |A_2|$ and $|A_i| \leq m$ for $i \in \{1, 2\}$.*

Proof: For $s \in \{1, 2\}$, set $m_s := \{i \in \{1, \dots, m\} \mid \lambda_i = s\}$. We know that $|A_1| + |A_2| = m_1 + 2m_2 = m + m_2$. We deduce that $|A_1| \leq m_2$ and $|A_2| \leq m_2$. This ensures that the following greedy procedure is valid. For every color i with $\lambda_i = 2$, we color one vertex in A_1 and one vertex in A_2 with i . After that, it remains to assign arbitrarily the m_1 colors of multiplicity 1 to the m_1 non-colored vertices. \square

Lemma 9 *Let $H := P(H_1, H_2)$ be a graph represented by a construction subtree of T . Assume that $\text{width}(H_i) \leq k - 2$ for $i \in \{1, 2\}$. Then $\text{width}(H) \leq k - 2$.*

Proof: We proceed by contradiction. Let H be a minimal counter-example. By Lemmas 7 and 8, we know that $\text{width}(H) = k + \mu$ for some positive integer μ .

Let a and b be the poles of H . We first prove that every component U of $H - \{a, b\}$ has at least $\mu + 2$ vertices. Indeed, since the root n of the construction subtree representing H is in normal form, the node n is a parallel node and the subgraph induced by $U \cup \{a, b\}$, from which we remove the edge ab if it is present, is represented by a children of n , so $H' := H - U$ is represented by a construction subtree of T . If moreover $|U| \leq \mu + 1$, then H' has width at least $k - 1$, thereby contradicting the minimality of H . In particular, $\text{width}(H_i) \geq \mu + 2 \geq 3$ for $i \in \{1, 2\}$, so $k \geq 5$.

Assume for the time being that neither a nor b dominates $\text{Inner}(H)$. Because every component of $H - \{a, b\}$ has at least three vertices, Lemma 5 implies that each of $\text{Inner}(H_1)$ and $\text{Inner}(H_2)$ is dominated by either a or b . Consequently, we may assume that a dominates $\text{Inner}(H_1)$ but not $\text{Inner}(H_2)$ and b dominates $\text{Inner}(H_2)$ but not $\text{Inner}(H_1)$. Let $u_1 \in \text{Inner}(H_1)$ and $u_2 \in \text{Inner}(H_2)$ be non-neighbors of b and a , respectively. We distinguish two cases depending on the value of μ .

First case: $\mu \leq 2$. Let F be the graph $G - \text{Inner}(H)$ to which we add a crystal $C(\mu)$ with poles a and b (keeping exactly one edge between a and b should G already contain one). Note that F is K_4 -minor-free, because H already contains a path between a and b and the vertices of F that do not belong to G have degree 0 in $F - \{a, b\}$. Let v_1, \dots, v_μ be the inner vertices of this new crystal. Note that $|V(G)| - |V(F)| = k$. Since $d_H(a) \geq \text{width}(H_1) \geq \mu + 2$ and similarly $d_H(b) \geq \mu + 2$, the graph F has maximum degree at most Δ .

By the minimality of G there is an equitable k -coloring α of F . Note that the restriction of α to $V(G) \setminus \text{Inner}(H)$ is also a proper partial coloring of G . To equitably color G , it suffices to extend this partial coloring to a proper

coloring β of G such that the multiset $\{\beta(v) \mid v \in \text{Inner}(H)\}$ equals the multiset $C := \{1, \dots, k\} \cup \{\alpha(v_i) \mid 1 \leq i \leq \mu\}$.

The colors $\alpha(a)$ and $\alpha(b)$ both have multiplicity exactly 1 in C and the maximal multiplicity in C is at most $\mu + 1$. We set $\beta(u_1) := \alpha(b)$ and $\beta(u_2) := \alpha(a)$.

If the maximal multiplicity in C is 2, then Observation 9 ensures that we can properly assign the $k - 2$ remaining colors since each of H_1 and H_2 has at most $k - 3$ non-colored vertices. This yields an equitable k -coloring of G , which is a contradiction.

If the maximal multiplicity in C is 3, then $\mu = 2$, so $\text{width}(H_1) \geq \mu + 2 \geq 4$. It follows then that $\text{Inner}(H_1) \setminus \{u_1\}$ contains an independent set $\{v_1, v_2\}$ of size 2. Indeed, otherwise $\text{Inner}(H_1) \setminus \{u_1\}$ would be a clique of size at least 3, which with a would induce a copy of K_4 in G . Since $\text{width}(H_2) \geq \mu + 2 \geq 4$, there exists a vertex v_3 in $\text{Inner}(H_2) \setminus \{u_2\}$. We color v_1, v_2 and v_3 with the (unique) color of multiplicity 3 in C . Again, observation 9 ensures that we can properly assign the $k - 3$ remaining colors since each of H_1 and H_2 has at most $k - 4$ non-colored vertices.

Second case: $\mu \geq 3$. Let F be the graph $G - \text{Inner}(H)$ to which we add the edge ab if it is not already present. By the minimality of G there is an equitable k -coloring α of F . To equitably color G , it suffices to extend α to a proper coloring of G such that the multiset $\{\alpha(v) \mid v \in \text{Inner}(H)\}$ equals the multiset $C := \{1, \dots, k, 1, \dots, \mu\}$.

As $\mu \geq 3$, every component of $H - \{a, b\}$ has at least $\mu + 2 \geq 5$ vertices. Consequently, a has two non-adjacent non-neighbors w_2 and w'_2 in $\text{Inner}(H_2)$. To see this, consider a component U of $\text{Inner}(H_2)$. By Lemma 2, the set U contains at most one neighbor of a . It follows that $|U \setminus N(a)| \geq 3$, which gives the announced property since by Corollary 7 the set U induces a tree in G . One proves similarly that b has two non-adjacent non-neighbors w_1 and w'_1 in $\text{Inner}(H_1)$. We set $\alpha(w_2) := \alpha(a)$, $\alpha(w_1) := \alpha(b)$. If $\alpha(a)$ has multiplicity two in C , that is, if $1 \leq \alpha(a) \leq \mu$, then we additionally set $\alpha(w'_2) := \alpha(a)$. Similarly, if $\alpha(b)$ has multiplicity two in C , then we additionally set $\alpha(w'_1) := \alpha(b)$. After this, each of H_1 and H_2 has at most $k - 3$ non-colored vertices. By Observation 9, we can extend this coloring using the $k - 2$ remaining colors in C .

From now on, we assume that a dominates $\text{Inner}(H)$. Set $F := G - (\text{Inner}(H) \cup \{a\})$. By the minimality of G there is an equitable k -coloring α of F . To equitably color G , it suffices to extend α to a proper coloring of G such that the multiset $\{\alpha(v) \mid v \in \text{Inner}(H) \cup \{a\}\}$ equals the multiset $C := \{1, \dots, k + \mu + 1\}$, where integers are reduced modulo k . Note that $k + \mu + 1 \leq \text{width}(H_1) + \text{width}(H_2) + 1 < 2k$ so every color has multiplicity either 1 or 2 in C .

The vertex a has at most $k - 3 - \mu$ colored neighbors. There are $k - 1 - \mu$ colors with multiplicity one in C . Consequently, it is possible to color a with a color of multiplicity one that is different from $\alpha(b)$.

We now place the color $\alpha(b)$. We know that $\text{width}(H) \geq k + \mu \geq 6$. By Lemma 2, and since each component of $H \setminus \{a, b\}$ has size at least $\mu + 2 \geq 3$, the vertex b has at least one non-neighbor in each of H_1 and H_2 . We color a number

of these non-neighbors equal to the multiplicity of $\alpha(b)$ in C (which is either 1 or 2) using the color $\alpha(b)$. Observation 9 then ensures that we can obtain an equitable coloring with the $k - 2$ remaining colors. \square

Lemma 10 *Let $H := S(H_1, H_2)$ be a graph represented by a construction subtree of T . Assume that $\text{width}(H_i) \leq k - 2$ for $i \in \{1, 2\}$. Then $\text{width}(H) \leq k - 2$.*

Proof: Suppose, on the contrary, that H contradicts the statement. Subject to this, we choose H to have as few vertices as possible. We may assume that $\text{width}(H) \geq k + 1$ by Lemmas 7 and 8. Let b be the common pole of H_1 and H_2 and let a and c be the other poles of H_1 and H_2 , respectively.

Case 1: For each $i \in \{1, 2\}$, the subgraph of G induced by $\text{Inner}(H_i)$ contains an independent set $\{u_i^1, u_i^2\}$ of size 2. Let F be the graph $G - \text{Inner}(H)$ to which we add the edge ac if it is not already present. By the minimality of G there is an equitable k -coloring α of F , which we aim to extend to G such that the multiset $\{\alpha(v) \mid v \in \text{Inner}(H)\}$ equals the multiset $C := \{1, \dots, \text{width}(H)\}$, where each integer is reduced modulo k .

We know that $\text{width}(H) \leq \text{width}(H_1) + \text{width}(H_2) + 1 \leq 2k - 3$. It follows that there is a color $\gamma \in \{1, \dots, k\} \setminus \{\alpha(a), \alpha(c)\}$ of multiplicity one in C . We set $\alpha(b) := \gamma$, $\alpha(u_1^1) := \alpha(c)$ and $\alpha(u_2^1) := \alpha(a)$, and possibly $\alpha(u_1^2) := \alpha(c)$ if $\alpha(c)$ has multiplicity two in C and $\alpha(u_2^2) := \alpha(a)$ if $\alpha(a)$ has multiplicity two in C . For each $i \in \{1, 2\}$, the subgraph H_i has at most $k - 3$ non-colored vertices left, so by Observation 9 it is possible to extend the coloring using the $k - 3$ remaining colors with the corresponding multiplicities.

Case 2: $\text{Inner}(H_1)$ induces a clique. We know that

$$\text{width}(H_1) \geq \text{width}(H) - \text{width}(H_2) - 1 \geq k + 1 - (k - 2) - 1 \geq 2.$$

By Corollary 7, $\text{Inner}(H_1)$ is a forest, so $\text{width}(H_1) = 2$. It forces moreover $\text{width}(H_2)$ to be $k - 2$. This in particular implies that $k \geq 4$. Observe that the minimality of H ensures that each of the poles a and c has at least two neighbors in H .

Let d and e be the inner vertices of H_1 . We define F to be the graph $G - (\text{Inner}(H_1) \cup \text{Inner}(H_2))$ to which we add the edges ab , bc and ac if not already present. Note that the graph thus obtained still has maximum degree at most Δ . By the minimality of G there is an equitable k -coloring α of F .

It remains to deduce an equitable k -coloring of G . To do so, we recolor b with a color γ different from $\alpha(a)$, from $\alpha(b)$ and from $\alpha(c)$, which is possible as $k \geq 4$. Next we color d with $\alpha(b)$ and e with $\alpha(c)$. It now suffices to distribute arbitrarily the colors in $\{1, \dots, k\} \setminus \{\gamma, \alpha(c)\}$ to the vertices in $\text{Inner}(H_2)$. \square

We are now ready to conclude.

Proof of Theorem 4: A direct induction on the tree T using Lemmas 9 and 10 shows that G has at most $k - 2$ inner vertices. This contradicts our assumption that $|V(G)| > k$, thereby finishing the proof of Theorem 4. \square

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