

Improved Bounds for Track Numbers of Planar Graphs

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Abstract

A *track layout* of a graph consists of a vertex coloring and a total order of each color class, such that no two edges cross between any two color classes. The *track number* of a graph is the minimum number of colors required by a track layout of the graph.

This paper improves lower and upper bounds on the track number of several families of planar graphs. We prove that every planar graph has track number at most 225 and every planar 3-tree has track number at most 25. Then we show that there exist outerplanar graphs whose track number is 5, which leads to the best known lower bound of 8 for planar graphs. Finally, we investigate leveled planar graphs and tighten bounds on the track number of weakly leveled graphs, Halin graphs, and X-trees.

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1 Introduction

A *track layout* of a graph is a partition of its vertices into sequences, called *tracks*, such that the vertices in each sequence form an independent set and no two edges between a pair of tracks cross each other. The *track number* of a graph is the minimum number of tracks in a track layout. Track layouts are formally introduced by Dujmović, Morin, and Wood [8], although similar concepts are implicitly studied in several earlier works [4, 11, 12]. An original motivation for studying track layouts is their connection with the existence of low-volume three-dimensional graph drawings: A graph with n vertices has a three-dimensional straight-line drawing in a grid of size $\mathcal{O}(1) \times \mathcal{O}(1) \times \mathcal{O}(n)$ if and only if it has track number $\mathcal{O}(1)$ [8, 9].

Track layouts are closely related to other models of linear graph layouts, specifically, stack and queue layouts. A *stack* layout of a graph consists of a linear order on the vertices and a partition of its edges so that no edges in a single part cross; that is, there are no edges (u, v) and (x, y) in the same part with $u < x < v < y$. A *queue* layout is defined similarly except no two edges in the same part *nest*; that is, there are no edges (u, v) and (x, y) in a part with $u < x < y < v$. The minimum number of parts needed in a stack (queue) layout of a graph is called its *stack number* (*queue number*) [12, 13]. A major result in the field is that track number is tied to queue number in a sense that one is bounded by a function of the other [9]. In particular, every t -track graph has a $(t - 1)$ -queue layout, and every q -queue graph has track number at most $4q \cdot 4q^{(2q-1)(4q-1)}$. The relationship between stack and track layouts is not that prominent but it is known that stack number is bounded by track number for bipartite graphs [9]; whether the reverse is true is an open question. Therefore, a study on track layouts may shed light on the relationship between the linear graph layouts.

In this paper we investigate lower and upper bounds on the track number of various families of planar graphs. Table 1 summarizes new and existing bounds described in the literature. A recent breakthrough result by Dujmović, Joret, Micek, Morin, Ueckerdt, and Wood [7] implies that all planar graphs have bounded track number (in fact, the result extends to every proper minor-closed class of graphs). Their analysis leads to a very large constant, as it is based on a fairly generic technique. We provide an alternative proof resulting in the upper bound of 225 for the track number of planar graphs (Section 4.1). An important ingredient of our construction is an improved upper bound of 25 for the track number of planar 3-trees (Section 3). For the lower bounds, we find an outerplanar graph that requires 5 tracks, which is worst-case optimal. This resolves an open question posed in [9] and provides the best lower bound of 8 on the track number of general planar graphs (Section 4.2).

Finally in Section 5, we study track layouts of (weakly) leveled planar graphs, which are the graphs with planar leveled drawings having no dummy vertices. This is a well studied family of planar graphs, as it is related to layered graph drawing and 1-queue layouts [12]; refer to Section 5 for a definition. We prove that the existing upper bound of 6 is worst-case optimal for the class of graphs,

Table 1: Track numbers of various families of planar graphs

Graph class	Upper bound				Lower bound			
	Old	Ref.	New	Ref.	Old	Ref.	New	Ref.
tree	3	[11]			3	[11]		
outerplanar	5	[9]			4	[9]	5	[Thm. 3]
series-parallel	15	[4]			6	[8]		
planar 3-tree	4,000	[1]	25	[Thm. 1]	6	[8]	8	[Cor. 1]
planar	461,184,080	[7]	225	[Thm. 2]	7	[9]	8	[Cor. 1]
X-tree	6	[2]	5	[Thm. 4]	3	[4]	5	[Thm. 5]
Halin	6	[2]			3	[9]	5	[Thm. 5]
weakly leveled	6	[2]			3	[2]	6	[Thm. 5]

while certain subfamilies (for example, X-trees) admit a layout on 5 tracks. The results close the gaps between upper and lower bounds on the track numbers for the subclasses of planar graphs. Our lower bounds in the section rely on computational experiments using a SAT formulation of the track layout problem (Section 5.2). We conclude the paper in Section 6 with possible future research directions and open problems.

2 Preliminaries

In this section we introduce necessary definitions and recall some known results about track layouts. Throughout the paper, $G = (V(G), E(G))$ is a simple undirected graph with $n = |V(G)|$ vertices and $m = |E(G)|$ edges.

Track Layouts Let $\{V_i : 1 \leq i \leq t\}$ be a partition of V such that for every edge $(u, v) \in E$, if $u \in V_i$ and $v \in V_j$, then $i \neq j$. Suppose that $<_i$ is a total order of the vertices in V_i . Then the ordered set $(V_i, <_i)$ is called a *track* and the partition is called a *t-track assignment* of G . An *X-crossing* in a track assignment consists of two edges, (u, v) and (x, y) , such that u and x are on the same track V_i with $u <_i x$, and v and y are on a different track V_j with $y <_j v$. A *t-track layout* of graph G , denoted $\mathcal{T}(G)$, is a *t-track assignment* with no X-crossings, and *track number*, $\text{tn}(G)$, is the minimum t such that G has a *t-track layout*. In particular, 1-track graphs have no edges, and 2-track graphs are the forests of caterpillars.

Some authors consider a relaxed definition of the concept, called *improper track layouts*, in which edges between consecutive vertices in a track are allowed [6, 11]. It can be easily seen that the tracks of such a layout can be doubled to obtain a proper track layout [8]. Thus, every graph with improper track number t has (proper) track number at most $2t$. In Section 5, we show that the upper bound can be smaller than $2t$ for some subclasses of graphs. In this paper we study only proper track layouts.

A basic result on track layouts is a “wrapping” lemma, which is due to

Dujmović et al. [9] (which in turn is based on the ideas of Felsner et al. [11]). Consider a track assignment whose index set is D -dimensional. That is, let $\{V_{i_1, \dots, i_D} : 1 \leq i_1 \text{ and } 1 \leq i_d \leq b_d \text{ for } 2 \leq d \leq D\}$ be a track layout of a graph G . Define the *partial span* of an edge $(u, v) \in E$ with $u \in V_{i_1, \dots, i_D}$ and $v \in V_{j_1, \dots, j_D}$ to be $|i_1 - j_1|$. The following lemma describes how to modify the track layout of G with possibly many tracks into a layout whose track number is bounded by a function of b_2, \dots, b_D and the maximum partial span.

Lemma 1 (Dujmović et al. [9]) *Let $\{V_{i_1, \dots, i_D} : 1 \leq i_1 \text{ and } 1 \leq i_d \leq b_d \text{ for } 2 \leq d \leq D\}$ be a track layout of a graph G with maximum partial span $s \geq 1$. Then*

$$\text{tn}(G) \leq (2s + 1) \cdot \prod_{2 \leq d \leq D} b_d$$

We stress that Lemma 1 can be applied for track layouts whose index set is one-dimensional. In that case, $\text{tn}(G) \leq 2s + 1$, where s is the maximum span of an edge for the single dimension.

Treewidth and Tree-Partitions A *tree-decomposition* of a graph G represents the vertices of G as subtrees of a tree, in such a way that the vertices are adjacent if and only if the corresponding subtrees intersect. The *width* of a tree-decomposition is one less than the maximum size of a set of mutually intersecting subtrees, and the *treewidth* of G is the minimum width of a tree-decomposition of G . For a fixed integer $k \geq 1$, a *k -tree* is a maximal graph of treewidth k , such that no more edges can be added without increasing its treewidth. Alternatively, a k -tree is defined recursively as follows: A k -clique is a k -tree, and the graph obtained from a k -tree by adding a new vertex adjacent to every vertex of a k -clique is also a k -tree. A subgraph of a k -tree is called a *partial k -tree*. In our proofs we do not directly use a tree-decomposition of a graph but utilize a related concept, a tree-partition, which is defined next.

Given a graph G , a *tree-partition* of G is a pair $(T, \{T_x : x \in V(T)\})$ consisting of a tree T and a partition of V into sets $\{T_x : x \in V(T)\}$, such that for every edge $(u, v) \in E$ one of the following holds: (i) $u, v \in T_x$ for some $x \in V(T)$, or (ii) there is an edge (x, y) of T with $u \in T_x$ and $v \in T_y$. The vertices of T are called the *nodes* and the sets $T_x, x \in V(T)$ are called the *bags* of the tree-partition.

The following well-known result provides a tree-partition of a k -tree.

Lemma 2 (Dujmović et al. [8]) *There exists a rooted tree-partition $(T, \{T_x : x \in V(T)\})$ of a k -tree G such that*

- *for every node x of T , the subgraph of G induced by the vertices of T_x is a connected partial $(k - 1)$ -tree;*
- *for every non-root node x of T , if y is a parent node of x in T , then the set of vertices in T_y having a neighbor in T_x forms a clique of size k in G .*

Let us describe how one can obtain a tree-partition of a k -tree G as in Lemma 2. Fix an arbitrary vertex $r \in V$ and perform a breadth-first search in G starting from r . For every $d \geq 0$ and every connected component induced by the vertices of G at distance d from the root, create a node of T associated with a bag containing the vertices of the component. Two nodes are adjacent if the vertices of the corresponding bags are joined by at least one edge of G . Dujmović et al. [8] show that the constructed graph T is indeed a tree, and that the vertices of each bag form a connected subgraph of a $(k - 1)$ -tree.

Layerings and H -Partitions A generalization of a tree-partition is the notion of an H -partition. An H -partition of a graph G is a partition of $V(G)$ into disjoint *bags*, $\{A_x : x \in V(H)\}$ indexed by the vertices of H , such that for every edge $(u, v) \in E(G)$ one of the following holds: (i) $u, v \in A_x$ for some $x \in V(H)$, or (ii) there is an edge $(x, y) \in E(H)$ with $u \in A_x$ and $v \in A_y$. In the former case, (u, v) is an *intra-bag* edge and in the latter case, it is an *inter-bag* edge.

A layering of a graph $G = (V, E)$ is an ordered partition (V_0, V_1, \dots) of V such that for every edge $(v, w) \in E$, if $v \in V_i$ and $w \in V_j$, then $|i - j| \leq 1$. If r is a vertex in a connected graph G and $V_i = \{v \in V \mid \text{dist}_G(r, v) = i\}$ for all $i > 0$, then (V_0, V_1, \dots) is called a *BFS-layering* of G . The *layered width* of an H -partition of a graph G is the minimum integer ℓ such that for some layering (V_0, V_1, \dots) of G , we have $|A_x \cap V_i| \leq \ell$ for every bag A_x of the partition and every integer $i \geq 0$.

The following lemma is a key ingredient for our proof of the upper bound on the track number of planar graphs.

Lemma 3 (Dujmović et al. [7]) *Every planar graph G has an H -partition of layered width 3 such that H is planar and has treewidth at most 3. Moreover, there is such a partition for every BFS-layering of G .*

3 Planar Graphs of Bounded Treewidth

In this section, we study track numbers of planar graphs of bounded treewidth. Our primary goal is improving the existing upper bound for planar 3-trees.

The following notion is introduced by Dujmović et al. [8]; it is also implicit in the work of Di Giacomo et al. [4]. Let $\{V_i : 1 \leq i \leq t\}$ be a t -track layout of a k -tree, G_k . We say that a clique C_1 of G_k precedes a clique C_2 of G_k with

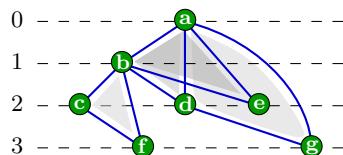


Figure 1: A 1-clique-colorable 4-track layout of a set of 3-cliques. A nice order \prec is defined as follows: $\langle b, c, f \rangle \prec \langle a, b, d \rangle \prec \langle a, d, g \rangle \prec \langle a, b, e \rangle$.

respect to the track layout if for every $1 \leq i \leq t$ and for all $u \in V_i \cap C_1$ and $w \in V_i \cap C_2$, it holds that $u \leq_i w$; we denote the relation by $C_1 \prec C_2$. Let $S = \{C_1, \dots, C_{|S|}\}$ be a set of maximal cliques of G_k . We say that S is *nicely ordered* if \prec is a total order on S , that is, $C_i \prec C_j$ for all $1 \leq i < j \leq |S|$. Finally, for a track layout of G_k , we say that a given set of maximal cliques is *c-colorable* if the set can be partitioned into c nicely ordered subsets. Such a track layout is called *c-clique-colorable*; see Figure 1 for an illustration.

Lemma 4 *The following holds:*

- (a) *Every path admits a 1-clique-colorable 2-track layout.*
- (b) *Every tree admits a 2-clique-colorable 3-track layout.*
- (c) *Every outerplanar graph admits a 2-clique-colorable 5-track layout.*

Proof: Claim (a) of the lemma is straightforward, as every 2-track layout of a path is 1-clique-colorable. Now we prove (b). Consider a plane drawing of a tree, T , such that the vertices having the same distance from the root are drawn on the same horizontal line; see Figure 2a. Let $x(v)$ and $y(v)$ denote x and y coordinates of a vertex $v \in V(T)$. The drawing corresponds to a track layout with maximum span 1, which by Lemma 1 can be converted into a 3-track layout by assigning $track(v) = y(v) \pmod{3}$ for every vertex v . The maximal cliques in the graph are edges of T . We partition the edges into $S_1 = \{(u, v) \in E(T) : track(u) = 1 \text{ or } track(v) = 1\}$ and $S_2 = E(T) \setminus S_1$. It is easy to verify that the two sets of edges are nicely ordered with respect to the 3-track layout.

In order to prove (c), we utilize a 5-track layout of an outerplanar graph suggested by Dujmović et al. [9]; see also [1]. They prove that every maximal outerplanar graph, G , has a straight-line outerplanar drawing in which vertex coordinates are integers, and the absolute value of the difference of the y -coordinates of the endvertices of each edge of G is either one or two; see Figure 2b. That is, $1 \leq |y(v) - y(u)| \leq 2$ for all $(u, v) \in E$. Such a drawing defines a track layout in which vertices with the same y -coordinates form a track and the ordering of the vertices within each track is implied by the drawing. The layout may have many tracks but maximum edge span is 2. By Lemma 1, the layout is wrapped onto 5 tracks by assigning $track(v) = y(v) \pmod{5}$ and ordering vertices within a track lexicographically by $(\lfloor y/5 \rfloor, x)$.

The maximal cliques in G are triangular faces. Denote a face containing vertices u, v, w by $\langle u, v, w \rangle$ and the set of all faces in G by F . We partition F into two sets $S_1 = \{\langle u, v, w \rangle \in F : track(u) = 2 \text{ or } track(v) = 2 \text{ or } track(w) = 2\}$ and $S_2 = F \setminus S_1$; see Figure 2b in which members of S_2 are shaded. All the faces of S_1 contain a vertex in track 2; choose the leftmost (having smallest x -coordinate) such vertex v^* with $track(v^*) = 2$. The faces containing v^* can be nicely ordered with respect to the track layout, as the drawing is planar. Removing v^* from the layout and applying the same argument for the remaining faces, yields a nice order of S_1 . An analogous procedure can be utilized to construct a nice order of S_2 , since all those faces contain a vertex in track 0 and a vertex in track 4. \square

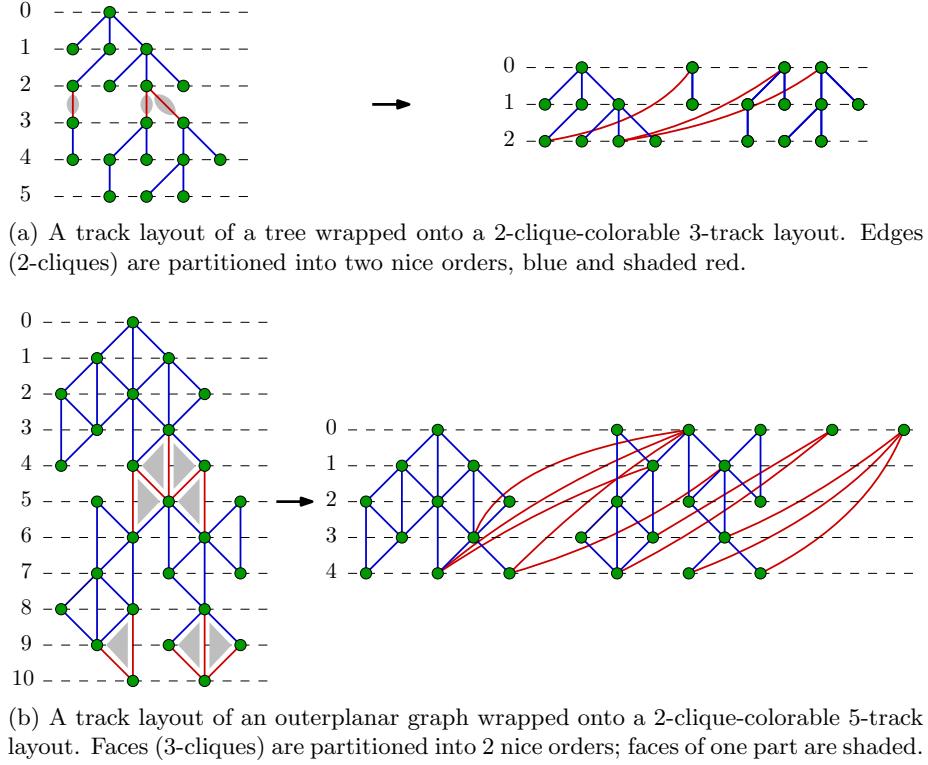


Figure 2: An illustration for Lemma 4

Notice that the claims of Lemma 4 are tight in terms of clique-colorability: There is no constant t such that every tree admits a 1-clique-colorable t -track layout. The next lemma shows how to use clique-colorable track layouts for graphs of bounded treewidth.

Lemma 5 *Assume that every k -tree admits a c -clique-colorable t -track layout. Then every $(k+1)$ -tree admits a layout on $t \cdot (2c+1)$ tracks.*

Proof: Let $(T, \{T_x : x \in V(T)\})$ be a tree-partition of a $(k+1)$ -tree, G_{k+1} , given by Lemma 2. In order to construct a desired track layout of G_{k+1} , consider a vertex $v \in V(G_{k+1})$ that belongs to a bag $x \in V(T)$ in the tree-partition. The track of v is defined by two indices, (i_v, j_v) , where i_v is derived from a track layout of T_x and j_v is derived from a certain track layout of the tree T . Next we define i_v and j_v .

- By Lemma 2, vertices of every bag of the tree-partition form a connected partial k -tree, which is a subgraph of a k -tree [3]. Thus, T_x admits a c -clique-colorable t -track layout, which we denote by $\mathcal{T}(T_x)$. The first index, i_v , is the track of v in $\mathcal{T}(T_x)$. Clearly the index ranges from 1 to t .

- Consider a parent node, $y \in V(T)$, of node x in the tree-partition. By Lemma 2, vertices of T_y adjacent to a vertex of T_x form a maximal clique, which we call the *parent clique* of x . The vertices of T_y form a partial k -tree, which by the assumption of the lemma admits a c -clique-colorable t -track layout, $\mathcal{T}(T_y)$. Thus, the parent clique of x has an assigned color, $c(x)$, ranging from 1 to c , and cliques with the same color are nicely ordered in $\mathcal{T}(T_y)$.

We layout T on (possibly many) tracks such that, for every parent node y , its child nodes are on c consecutive tracks. Formally, if x is a child of y and the corresponding parent clique of x has color $1 \leq c(x) \leq c$, then $\text{track}(x) = \text{track}(y) + c(x)$. The order of child nodes having the same color follows the nice order of the corresponding parent cliques. The constructed track layout of T is denoted $\mathcal{T}(T)$.

We assign j_v to be the track of node x in $\mathcal{T}(T)$. The index can be as large as $\Omega(|V(T)|)$ but the maximum span of an edge of $E(T)$ is c .

Now we define the order of the vertices of G_{k+1} within the same track. If two vertices, u and v , belong to the same bag $x \in V(T)$ in the tree-partition, their relative order is inherited from track layout $\mathcal{T}(T_x)$. If the vertices are in different bags, that is, $u \in T_x, v \in T_y$ for some $x \in V(T), y \in V(T)$, then the order is dictated by the order of nodes x and y in track layout $\mathcal{T}(T)$.

Next we show that the constructed track assignment has no X-crossings. Intra-bag edges do not form X-crossings by the assumption of the lemma. Consider two inter-bag edges, $(u_1, v_1) \in E(G_{k+1})$ and $(u_2, v_2) \in E(G_{k+1})$. Since there are no crossings in $\mathcal{T}(T)$, the inter-bag edges mapped to edges of T without a common parent are not in an X-crossing. Thus we may assume that $u_1 \in T_p, u_2 \in T_p, v_1 \in T_x, v_2 \in T_y$ for some nodes $p, x, y \in V(T)$ such that p is a parent of x and y . If parent cliques of x and y are of different colors, then by construction of $\mathcal{T}(T)$, vertices v_1 and v_2 are in different tracks. If parent cliques of x and y are of the same color, then the order between v_1 and v_2 is consistent with the order between u_1 and u_2 , since the cliques are nicely ordered in $\mathcal{T}(T_p)$. Therefore, an X-crossing between (u_1, v_1) and (u_2, v_2) is impossible.

Finally, we apply Lemma 1 for the constructed two-dimensional track layout with maximum partial span c to get the desired claim. \square

By combining Lemma 5 with Lemma 4, we can get an improved upper bound for the track number of planar 3-trees.

Theorem 1 *The track number of a planar 3-tree is at most 25.*

Proof: First we build a tree-partition of a planar 3-tree G as in Lemma 2. To this end, as mentioned in Section 2, pick a root $r \in V$ and perform a breadth-first search in G starting from r . Bags of the tree-partition are formed by connected components induced by the vertices of G at the same distance from the root.

Since G is planar, vertices at the same distance from the root induce an outerplanar graph [1]. Hence, the vertices of each bag of the tree-partition form

a connected outerplanar graph, which by Lemma 4 admits a 2-clique-colorable 5-track layout. Applying the arguments of Lemma 5 to a planar 3-tree G yields the claim of the theorem. \square

Note that a similar construction provides an upper bound of 15 for planar 2-trees (series-parallel graphs), as every bag of a corresponding tree-partition induce a tree, which admits a 2-clique-colorable 3-track layout. The same upper bound is already known [4].

4 General Planar Graphs

In this section we investigate upper and lower bounds on the track number of general planar graphs.

4.1 An Upper Bound

Recently Dujmović et al. [7] used H -partitions of bounded layered width to prove that the queue number of a planar graph is a constant. By analogy with their result, we show that the track number of a graph is bounded by a function of the track number of H and the layered width.

Lemma 6 *If a graph G has a layered H -partition of layered width ℓ , then G has track number at most $3\ell \cdot \text{tn}(H)$.*

Proof: Assume G has a layered H -partition of width ℓ , and suppose $\mathcal{T}(H)$ is a layout of H on $\text{tn}(H)$ tracks. To define a track assignment of G , $\mathcal{T}(G)$, consider a vertex $v \in V(G)$ that belongs to a bag $x \in V(H)$. The track of v is defined by three indices, (i_v, j_v, d_v) .

- The first index, i_v , is the track of x in track layout $\mathcal{T}(H)$; it ranges from 1 to $\text{tn}(H)$.
- The bag $x \in V(H)$ contains at most ℓ vertices in every layer. Label these vertices arbitrarily from 1 to ℓ , and assign the second index, j_v , to the label. Thus, $1 \leq j_v \leq \ell$.
- The last index, d_v , represents the layer of v in the given layered H -partition of G . Clearly, $d_v \geq 1$ and d_v is at most the number of layers in G , which can be as large as $\Omega(|V(G)|)$.

In order to complete the track assignment, we define the order of vertices in the same track. Notice that vertices of every bag, A_x for some $x \in V(H)$, are on different tracks defined by the second and the third indices of the track assignment. Therefore, only the vertices of G corresponding to different bags can belong to the same track. For those vertices, the order is inherited from the given track layout $\mathcal{T}(H)$. That is, $v < u$ in $\mathcal{T}(G)$ with $v \in A_x, u \in A_y$ if and only if $x < y$ in $\mathcal{T}(H)$; see Figure 3.

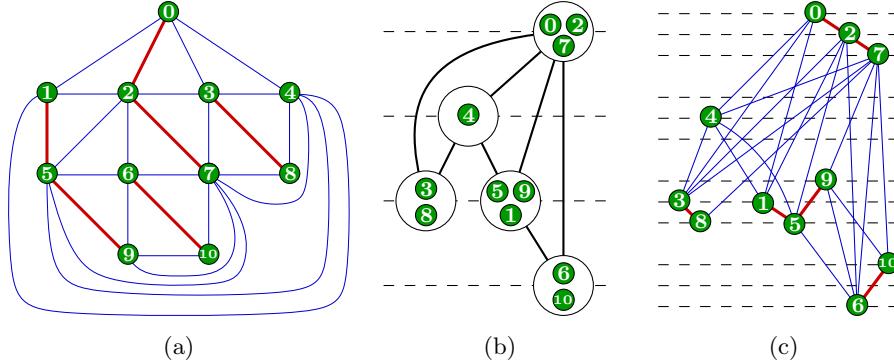


Figure 3: (a) A planar graph with a layered H -partition rooted at $v = 0$ with layered width $\ell = 1$; intra-bag edges are red. (b) A 4-track layout of H . (c) A 12-track layout of the graph constructed using Lemma 6.

Now we verify that $\mathcal{T}(G)$ is a valid track layout, that is, it contains no X-crossings. For a contradiction suppose that $(u_1, v_1) \in E(G)$ and $(u_2, v_2) \in E(G)$ form an X-crossing, that is, $track(u_1) = track(u_2)$, $track(v_1) = track(v_2)$ and $u_1 < u_2, v_2 < v_1$. Since all vertices of a bag are on different tracks, it follows that u_1 and u_2 belong to different bags and that v_1 and v_2 belong to different bags. Therefore, the two edges correspond either to an X-crossing in $\mathcal{T}(H)$ (if the two edges are intra-bag edges), or to an edge of H with both endpoints on the same track of $\mathcal{T}(H)$ (if one of the edges is an inter-bag edge). Both of the options violate the definition of $\mathcal{T}(H)$; hence, $\mathcal{T}(G)$ contains no X-crossings.

Finally, observe that the partial span in $\mathcal{T}(G)$ corresponding to the third dimension of the track assignment, d_v , is at most one, as it is based on a layering of G . By Lemma 1, the track layout can be wrapped onto $3\ell \cdot tn(H)$ tracks; Figure 3 illustrates the process. \square

Combining Lemma 6, Lemma 3, and Theorem 1, we get the following result.

Theorem 2 *The track number of a planar graph is at most 225.*

4.2 Lower Bounds

Dujmović et al. [9] show an outerplanar graph that requires 4 tracks and prove that every outerplanar graph has a 5-track layout. Our next result closes the gap between the lower and the upper bounds answering the question posed in [9].

Theorem 3 *The outerplanar graph in Figure 4a has track number 5.*

Before proving the theorem, we introduce two configurations that we use in the proof. The first configuration, illustrated in Figure 4b, is defined on four vertices forming a cycle. If $track(a) = 1$, $track(b) = track(c) = 2$, $track(d) = 3$, and $b < c$, then for every vertex v that is “inside” the quadrangle (that is,

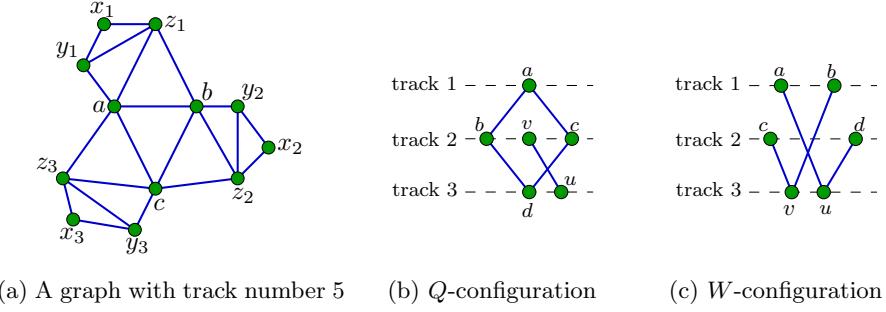


Figure 4: An illustration for Theorem 3

$\text{track}(v) = 2$ and $b < v < c$), its neighbor u is not on tracks 1 and 3. We call this a $Q(a, b < v < c, d)$ -*configuration* for vertex v . The second configuration, illustrated in Figure 4c, is defined on four vertices, a, b, c, d , with $\text{track}(a) = \text{track}(b) = 1$, $\text{track}(c) = \text{track}(d) = 2$ and $a < b, c < d$. If there exist two vertices u and v together with edges $(u, a), (u, d), (v, b), (v, c)$, then $\text{track}(u) \neq \text{track}(v)$. We call this a $W(a < b, c < d)$ -*configuration* for vertices u and v . We emphasize that for both configurations, the actual tracks of the vertices are irrelevant; it is only important which vertices share tracks.

Proof of Theorem 3: Assume that the graph in Figure 4a has a 4-track layout. Without loss of generality, we may assume that vertex a is on track 1, vertex b is on track 2, and vertex c is on track 3. Next we consider tracks of vertices z_1 , z_2 , and z_3 , and distinguish four cases depending on how many of the vertices are on track 4.

- None of z_1, z_2, z_3 are on track 4.

It is easy to see that $\text{track}(z_1) = 3$, $\text{track}(z_2) = 1$, and $\text{track}(z_3) = 2$; see Figure 5a. Assume without loss of generality that $b < z_3$, that is, vertex b precedes vertex z_3 on track 2. Then in order for edge (c, z_3) to avoid a crossing with edge (b, z_1) , vertex z_1 should precede c on track 3, that is, $z_1 < c$. Then a and z_2 form a $W(z_1 < c, b < z_3)$ -configuration, a contradiction.

- All of z_1, z_2, z_3 are on track 4.

Assume without loss of generality that $z_1 < z_2 < z_3$. We show that the graph in Figure 5b is not embeddable in four tracks.

Observe that z_2 's neighbors cannot be on track 1; otherwise, the edge from z_2 to the neighbor crosses one of the edges (a, z_1) or (a, z_3) . Therefore, $\text{track}(y_2) = 3$ and $\text{track}(x_2) = 2$. Notice that $y_2 < c$, as otherwise edges (y_2, z_2) and (c, z_3) cross. Then x_2 and b form a $W(z_1 < z_2, y_2 < c)$ -configuration, a contradiction.

- One of z_1, z_2, z_3 is on track 4; suppose $\text{track}(z_1) = 4$.

It is easy to see that $\text{track}(z_2) = 1$ and $\text{track}(z_3) = 2$. Assume without loss of generality that $z_3 < b$; it follows that $a < z_2$, as otherwise (b, z_2) and (a, z_3) cross. Next we prove that the graph in Figure 5c is not embeddable in four tracks. We distinguish two cases depending on the track of y_1 .

First assume $\text{track}(y_1) = 3$. It holds that $y_1 < c$, as otherwise (c, z_2) and (y_1, a) cross. Now it is impossible to assign a track for y_3 : If $\text{track}(y_3) = 1$, then y_3 and a form a $W(y_1 < c, z_3 < b)$ -configuration; If $\text{track}(y_3) = 4$, then y_3 and z_1 form a $W(y_1 < c, z_3 < b)$ -configuration.

Second assume $\text{track}(y_1) = 2$. It holds that $y_1 < b$, as otherwise (b, z_2) and (a, y_1) cross. If $z_3 < y_1 < b$, then y_1 forms a $Q(c, z_3 < b, a)$ -configuration. If $y_1 < z_3 < b$, then z_3 forms a $Q(z_1, y_1 < b, a)$ -configuration, a contradiction.

- Two of z_1, z_2, z_3 are on track 4. Suppose $\text{track}(z_1) = \text{track}(z_2) = 4$; thus, $\text{track}(z_3) = 2$.

Assume without loss of generality that $z_1 < z_2$. Next we prove that the graph in Figure 5d is not embeddable in four tracks. We distinguish two cases depending on the relative order of b and z_3 .

First assume $b < z_3$. Consider the track of y_2 . If $\text{track}(y_2) = 1$, then a and y_2 form a $W(z_1 < z_2, b < z_3)$ -configuration; thus, $\text{track}(y_2) = 3$. To avoid a crossing between edges (c, z_3) and (b, y_2) , we have $y_2 < c$. Now it is not possible to layout x_2 . If $\text{track}(x_2) = 1$, then a and x_2 form a $W(z_1 < z_2, y_2 < c)$ -configuration. Otherwise if $\text{track}(x_2) = 2$, then b and x_2 form a $W(z_1 < z_2, y_2 < c)$ -configuration.

Second assume $z_3 < b$. Consider the track of y_1 . If $\text{track}(y_1) = 3$, then $y_1 < c$, as otherwise (y_1, z_1) and (c, z_2) cross. Now it is impossible to layout y_3 , which forms and a $W(y_1 < c, z_3 < b)$ -configuration with vertex a if $\text{track}(y_3) = 1$, and it forms a $W(y_1 < c, z_3 < b)$ -configuration with vertex z_1 if $\text{track}(y_3) = 4$.

Thus, $\text{track}(y_1) = 2$. It follows that $y_1 < b$, as otherwise (y_1, z_1) and (b, z_2) cross. If $z_3 < y_1 < b$, then y_1 is in a $Q(a, z_3 < b, c)$ -configuration and its neighbor, x_1 , cannot be placed. Otherwise if $y_1 < z_3 < b$, then z_3 is in a $Q(a, y_1 < b, z_1)$ -configuration, and y_3 cannot be placed.

Therefore, the graph in Figure 4a is not embeddable in a 4-tracks, which concludes the proof of the theorem. \square

Dujmović et al. (Lemma 22 of [9]) show that, for every outerplanar graph H with track number $\text{tn}(H)$, there exists a planar graph whose track number is $\text{tn}(H) + 3$. We stress that the graph in their construction is a planar 3-tree. Therefore, we have the following improved lower bound for the track number of planar 3-trees.

Corollary 1 *There exists a planar 3-tree G with track number $\text{tn}(G) = 8$.*

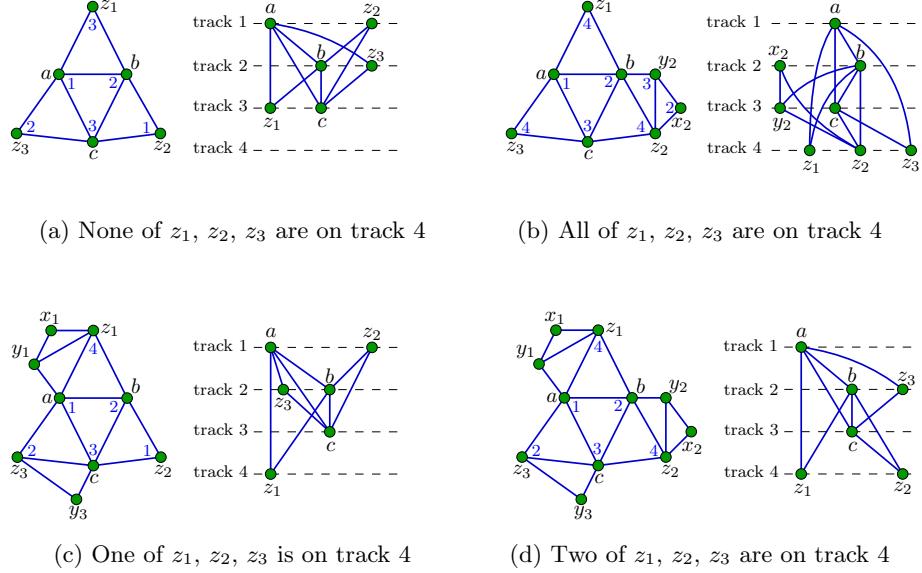


Figure 5: Different cases in the proof of Theorem 3. Track of a vertex is blue.

5 Other Subclasses of Planar Graphs

Track layouts of planar graphs are related to leveled planar graph drawings that were introduced by Heath and Rosenberg [12] in the context of queue layouts. A leveled planar drawing of a graph is a straight-line crossing-free drawing in the plane, such that the vertices are placed on a sequence of parallel lines (levels) and every edge joins vertices in two consecutive levels. A graph is *leveled planar* if it admits a leveled planar drawing. Bannister et al. [2] characterize the class of graphs by showing that a graph is leveled planar if and only if it is bipartite and admits a 3-track layout. In a relaxed definition of leveled drawings, edges between consecutive vertices on the same level are allowed; this leads to *weakly leveled planar* graphs. For graphs that have a weakly leveled planar drawing, Bannister et al. [2] show the upper bound of 6 for the track number, while leaving the question of the lower bound open. We answer the question by providing an example of a weakly leveled planar graph whose track number is 6.

Certain families of planar graphs are known to admit weakly leveled planar graphs; for example, Halin graphs (an embedded tree with no vertices of degree 2 whose leaves are connected by a cycle) and X-trees (a complete binary tree with extra edges connecting vertices of the same level). Although track numbers of the graphs have been investigated in several earlier works [2, 4, 6], the gaps between lower and upper bounds remain open. In the following, we present a Halin graph and an X-tree that require 5 tracks. In addition we provide an algorithm that constructs a 5-track layout for every X-tree, thus, closing the gap between lower and upper bounds of the track number of such graphs.

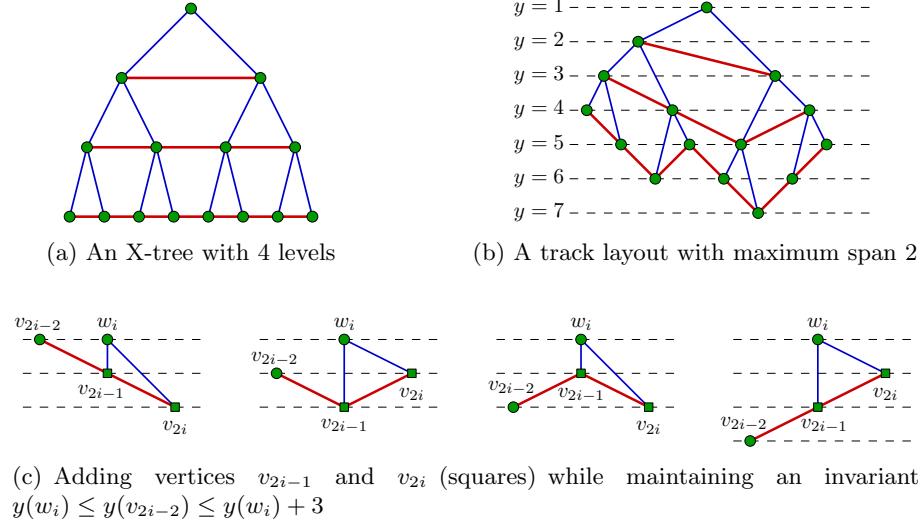


Figure 6: An illustration for Theorem 4: constructing a 5-track layout for X-trees

Our lower bound examples in the section rely on computational experiments. To this end, we propose a SAT formulation of the track layout problem, and share the source code of our implementation [14]. The formulation is simple-to-implement but efficient enough to find optimal track layouts of graphs with up to a few hundred of vertices in a reasonable amount of time.

5.1 An Upper Bound for X-trees

An *X-tree* is a complete binary tree with extra edges connecting vertices of the same level. Formally, if v_1, v_2, \dots, v_{2^d} are the vertices of level $d \geq 0$ in the tree in the left-to-right order, then the extra edges are (v_i, v_{i+1}) for $1 \leq i < 2^d$; see Figure 6a. Since X-trees admit a weakly leveled planar drawing, their track number is at most 6 [2, 4, 6]. Next we improve the upper bound to 5.

Theorem 4 *Every X-tree has a 5-track layout.*

Proof: We call the edges between the vertices of the same level in the X-tree *level edges*, while the remaining edges are *tree edges*. In order to construct a 5-track layout, we build a planar straight-line drawing of the graph such that every vertex v is laid out on an integer grid with coordinates $x(v) \in \mathbb{N}$ and $y(v) \in \mathbb{N}$. In the drawing we maintain a property that $|y(u) - y(v)| \leq 2$ for every edge (u, v) ; see Figure 6b. It is easy to see that such a drawing corresponds to a track layout with span 2 in which the vertices having equal y -coordinates are on the same track. By Lemma 1, the layout can be wrapped onto 5 tracks.

The drawing is built inductively on the level, $d \geq 0$, of a given X-tree using a hypothesis that every X-tree admits a planar straight-line drawing such that:

- $|y(u) - y(v)| = 1$ for every level edge (u, v) and $|y(u) - y(v)| \leq 2$ for every tree edge;
- for the level- d vertices in the tree, v_1, v_2, \dots, v_{2^d} , it holds that $x(v_1) < x(v_2) < \dots < x(v_{2^d})$; equivalently, the corresponding level edges form a strictly x -monotone polyline;
- every edge of the highest level, (u, v) , is on the boundary of the drawing; that is, it is visible from points $(x(u), +\infty)$ and $(x(v), +\infty)$.

The basis of the induction for $d \leq 1$ is trivial; see Figure 6b. In order to construct the drawing for an X-tree, G , for $d > 1$, we start with an inductively constructed X-tree, G' , of level $d - 1$. Denote the vertices of G of level $d - 1$ by $w_1, \dots, w_{2^{d-1}}$ and the vertices of level d by v_1, \dots, v_{2^d} . Observe that G can be constructed from G' by iteratively adding a pair of vertices, v_{2i-1} and v_{2i} , along with three edges (v_{2i-1}, v_{2i}) , (v_{2i-1}, w_i) , and (v_{2i}, w_i) , where $1 \leq i \leq 2^{d-1}$. Starting from a drawing of G' , we add the pairs of vertices while maintaining an invariant that $y(w_i) \leq y(v_{2i-2}) \leq y(w_i) + 3$ for all $2 \leq i < 2^{d-1}$.

Clearly, the invariant is initialized for $i = 2$ by setting $x(v_1) = 1, y(v_1) = y(w_1) + 1$ and $x(v_2) = 2, y(v_2) = y(w_1) + 2$. For $i > 2$, we distinguish the cases based on the y -coordinates of the previously placed vertices. As illustrated in Figure 6c, there are four cases:

- case (a) $y(v_{2i-2}) = y(w_i)$: assign $y(v_{2i-1}) = y(w_i) + 1, y(v_{2i}) = y(w_i) + 2$;
- case (b) $y(v_{2i-2}) = y(w_i) + 1$: assign $y(v_{2i-1}) = y(w_i) + 2, y(v_{2i}) = y(w_i) + 1$;
- case (c) $y(v_{2i-2}) = y(w_i) + 2$: assign $y(v_{2i-1}) = y(w_i) + 1, y(v_{2i}) = y(w_i) + 2$;
- case (d) $y(v_{2i-2}) = y(w_i) + 3$: assign $y(v_{2i-1}) = y(w_i) + 2, y(v_{2i}) = y(w_i) + 1$.

In all of the cases, assign $x(v_i) = i$ for $1 \leq i \leq 2^d$. One can easily verify that the desired invariants are maintained and the resulting drawing is planar. \square

5.2 Lower Bounds

To test whether a given graph $G = (V, E)$ admits a t -track layout, we formulate a Boolean Satisfiability Problem that has a solution if and only if G has a layout on t tracks. We introduce two sets of variables:

- a variable $\phi_q(v)$ for every vertex $v \in V$ and every track $1 \leq q \leq t$ indicating whether the vertex belongs to the track;
- a variable $\sigma(v, u)$ for every pair of vertices $v \in V, u \in V$ indicating whether v precedes u in the order, that is, $v < u$ for some track of the layout.

Every vertex is assigned to one track, which is ensured by the track assignment rules:

$$\phi_1(v) \vee \phi_2(v) \vee \dots \vee \phi_t(v) \quad \forall v \in V \quad \text{and} \quad \neg\phi_i(v) \vee \neg\phi_j(v) \quad \forall v \in V, 1 \leq i < j \leq t$$

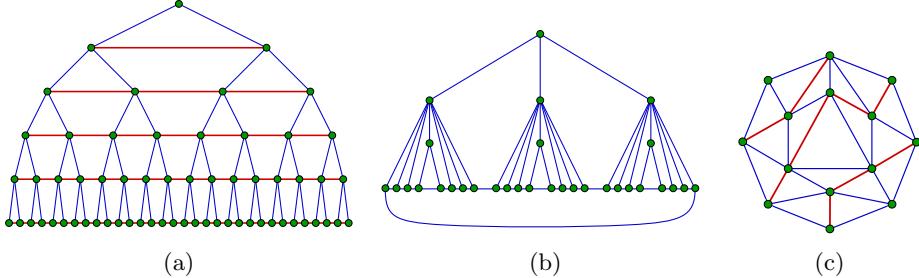


Figure 7: (a) An X-tree and (b) a Halin graph that require 5 tracks. (c) A weakly leveled planar graph that requires 6 tracks; level edges are red.

To guarantee a valid track assignment, we forbid adjacent vertices on the same track:

$$\neg\phi_i(v) \vee \neg\phi_i(u) \quad \forall (u, v) \in E, 1 \leq i \leq t$$

For the relative encoding of vertices, we ensure asymmetry and transitivity:

$$\sigma(v, u) \leftrightarrow \neg\sigma(u, v) \quad \forall \text{ distinct } u, v \in V$$

$$\sigma(v, u) \wedge \sigma(u, w) \rightarrow \sigma(v, w) \quad \forall \text{ distinct } u, v, w \in V$$

To forbid X-crossings among edges, recall that an X-crossing between edges $(u_1, v_1) \in E, (u_2, v_2) \in E$ occurs when $\text{track}(u_1) = \text{track}(u_2), \text{track}(v_1) = \text{track}(v_2)$ and $u_1 < u_2, v_1 > v_2$. This can be expressed by the following rules:

$$\phi_i(u_1) \wedge \phi_i(u_2) \wedge \phi_j(v_1) \wedge \phi_j(v_2) \rightarrow (\sigma(u_1, u_2) \wedge \sigma(v_1, v_2)) \vee (\sigma(u_2, u_1) \wedge \sigma(v_2, v_1))$$

$$\forall (u_1, v_1) \in E, (u_2, v_2) \in E \text{ such that } u_1 \neq u_2, v_1 \neq v_2 \text{ and } \forall 1 \leq i, j \leq t$$

The resulting CNF formula contains $\Theta(n^2)$ variables and $\Theta(n^3 + m^2t^2)$ clauses. Using a modern SAT solver, one can evaluate small and medium size instances (with up to a few hundred of vertices) within a reasonable time. For example, we computed optimal track layouts for all 977,526,957 maximal planar graphs having $n = 18$ vertices. In total the computation took 5000 machine-hours, and all the graphs turned out to have a t -track layout for some $4 \leq t \leq 7$. Larger graphs, such as one in Corollary 1, are solved within a few hours on a regular machine. Our implementation is available at [14].

Using the formulation, we identified examples of an X-tree and a Halin graph that require 5 tracks; see Figure 7a and 7b. In particular, X-trees of depth $d \leq 5$ admit a 4-track layout but X-trees with $d \geq 6$ have track number 5. Similarly, we found a weakly leveled planar graph with 14 vertices that has track number 6; see Figure 7c. Thus, the algorithm of Bannister et al. [2] for constructing 6-track layouts of weakly leveled planar graphs is worst-case optimal. The results are summarized as follows.

Theorem 5 *There exists an X-tree and a Halin graph with track number 5. There exists a weakly leveled planar graph with track number 6.*

6 Conclusions and Open Problems

In this paper we improved upper and lower bounds on the track number of several families of planar graphs. A natural future direction is to close the remaining gaps for graphs listed in Table 1. Next we discuss several open questions related to track layouts.

Our approach for building track layouts of planar graphs relies on a construction for graphs of bounded treewidth. To the best of our knowledge, the upper bound on the track number of k -trees is $(k+1)(2^{k+1}-2)^k$ [15], while the lower bound is only quadratic in k [8]. It seems unlikely that the existing upper bound is the right answer, and finding a polynomial or even an exponential $2^{\mathcal{O}(k)}$ bound would already be an exciting improvement.

Open Problem 1 *Improve the upper bound on the track number of k -trees.*

One way of attacking the above problem is tightening a gap between track and queue numbers of a graph, as the queue number of a k -tree is at most $2^k - 1$ [15]. As mentioned earlier, every t -track graph has a $(t-1)$ -queue layout. It is easy to see that this bound is worst-case optimal, that is, there exist t -track graphs that require $t-1$ queues in every layout. In the other direction, every q -queue graph has track number at most $4q \cdot 4q^{(2q-1)(4q-1)}$ [9]. Similarly, every q -queue graph with acyclic chromatic number c has track-number at most $c(2q)^{c-1}$ [8]. Recall that a vertex coloring is *acyclic* if there is no bichromatic cycle. These bounds can likely be improved. For example, 1-queue graphs always admit a 4-track layout but no better bound is known for the case $q \geq 2$.

Open Problem 2 *What is the largest track number of a q -queue graph?*

Finally, we would like to see some progress on *upward* track layouts [5, 10]. An *upward track layout* of a dag G is a track layout of the underlying undirected graph of G , such that the directed graph obtained from G by adding arcs between consecutive vertices in a track is acyclic. To the best of our knowledge, very little is known about this variant of layouts. For example, the upward track number of (directed) paths and caterpillars is known to be 3 [5], while the upward track number of (directed) trees is at most 5 [10]. Improving the bounds and investigating other classes of graphs is an interesting research direction.

Open Problem 3 *Investigate the upward track number of various families of graphs.*

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