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On Dispersability of Some Circulant Graphs

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Abstract. The matching book thickness of a graph is the least number of pages in a book embedding such that each page is a matching. A graph is dispersable if its matching book thickness equals its maximum degree. Minimum page matching book embeddings are given for bipartite and for most non-bipartite circulants contained in the (Harary) cube of a cycle and for various higher-powers.

1 Introduction

Dispersable graphs were introduced in [4], where it was conjectured that all regular bipartite graphs are dispersable. This was disproved by Alam et al. [1] who showed that the Gray and Folkman graphs, though regular bipartite, are not dispersable. These counterexamples are edge-transitive but not vertex-transitive. In [2], Alam et al. gave an infinite family of counterexamples to the claim and conjectured that *bipartite vertex-transitive graphs are dispersable.*

In this paper, the target graph families are circulant graphs (hence, vertex-transitive). Circulants (or "Harary graphs" [10]) are used in graph theory, computer science, network engineering, and dynamical systems (e.g., [5, 9, 12, 14, 26]). Terms are defined in the next section.

Matching book embeddings of bipartite circulants C are given where the page number is equal to the vertex degree $\Delta(C)$, supporting the conjecture [2]. Regular dispersable graphs are bipartite [21]. A nonbipartite circulant is *nearly dispersable* if it needs one extra page [21]. So far, all nonbipartite circulants have been nearly dispersable and we conjecture that *nonbipartite, vertex-transitive graphs are nearly dispersable.*

Previous results support both conjectures. For the complete bipartite graph and the hypercube, see [4]; for complete graphs and other bipartite graphs, see [21]. Cartesian products of even cycles are dispersable; even times odd cycles are nearly dispersable [17]; and short odd (length at most 5) and arbitrary odd cycles have nearly dispersable product [15]. Other classes of vertex-transitive graph that are known to be nearly dispersable include the product of two *arbitrary* cycles and of

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cycles with complete graphs, see [23, 25], and some products of bipartite and nonbipartite graphs [22]. See also [27] and §8. Some graphs which are *not* vertex transitive also are dispersable such as trees [21], Halin trees [24], and cubic planar bipartite graphs [2, 19].

To define good matching book embeddings for an infinite family of graphs, one needs to give both layout and coloring schemes: algorithms which produce the needed vertex order and edge-to-page assignment from the various integers that identify each graph in the family. Most of our families consist of circulants $C(n, S)$ with a fixed jump-length set S and with the number n of vertices reduced modulo 2 or 4. The coloring algorithms can either be static (as in tables based on modularity) or dynamic (as in prescriptions for Hamiltonian cycles or paths). See proofs of Theorems 1 and 3, resp.

It turns out, however, that for nonbipartite circulants, perfectly regular patterns almost never succeed and irregularity is forced. Irregular features appear in two different ways: local and global.

The local type of exception is involved in the "twist" (see §3) while the global type manifests as "sparseness" in many examples, where one nearly reaches the lower bound except for a "sparse" page with a small and structurally defined set of exceptional edges. Computer search [18, p 7] gives random vertex-order, while our vertex-orders and edge-to-page functions are quite regular. Nevertheless, the edges of the sparse page are irregularly distributed in a characteristic pattern for all parameter values with the same modularity.

Any strategy to achieve the minimum number of pages in a matching book embedding of a graph family, such as $C(n, S)$, based on regular layout and algorithms is a kind of "polymerization process" since almost all edges are placed in a repeated pattern. A polymer is a molecule composed of a sequence of many parts such as proteins composed of amino acids or RNA/DNA as a sequence of nucleotides. The sequence of parts may form a path or a cycle.

Here are four examples of *generalized polymerization*, a phenomenon that we believe deserves more thorough investigation. In each of these examples, a finite set of adjustments permits regularity for the arbitrarily large remainder.

(1) The coloring irregularities given in [18] for $C(2k+r, \{1, k\})$, $r \in \{0, 1, 2\}$, with up to 5 edges on the sparse page, allow a minimum page embedding.

(2) Layouts and page-partition of the Cartesian product of C_3 or C_5 with another cycle [15] use a "seed" that is an exceptional copy of a repeated motif.

(3) The twist used in §3 below for the case $C(2k, \{1, 3\})$ with a dispersable circulant and in §4 for $C(4k + 3, \{1, 3\})$ in the nearly dispersable case.

(4) Most of the matching book embeddings in this paper have a sparse page.

In contrast, *strict polymerization* is defined below to be an algorithmic procedure which puts together certain modular units with no adjustments.

The paper is organized as follows: §2 has definitions, §3 shows $C(n, \{1, 3\})$ is dispersable for n even, while §4 shows $C(n, \{1, 3\})$ is nearly dispersable (n.d.) for n odd. In §5 and §6, we prove $C(n, \{1, 2\})$ and $C(n, \{2, 3\})$ are n.d., and §7 shows $C(n, \{1, 2, 3\})$ is n.d. when n is odd or a multiple of 7 or 12 (so $\geq 64.3\,\%$ of the $C(n, \{1, 2, 3\})$ circulants); §7 also shows that $K_{2k} - kK_2$ and $K_{2k+1} - C_{2k+1}$ are n.d. The last section has applications and a discussion.

2 Definitions

Undefined terms are as in [11].

The **circulant graph** $C(n, S)$ of order n with **jump** set $S = \{i_1, \ldots, i_k\}$ is the graph on $[n] := \{1, \ldots, n\}$, where $j \in [n]$ is adjacent to $j + i_r$ (addition mod n), $r = 1, \ldots, k$ and $1 \leq i_1 < i_2 < \cdots < i_k \leq \lfloor n/2 \rfloor$, $k \geq 1$. A graph is **vertex-transitive** if for any two vertices, there is an isomorphism carrying one to the other. The k-th **Harary power** C_n^k of an order-n cycle C_n [11, p 14] is the circulant $C(n,[k])$, and any circulant of order n with maximum jump k is a vertex-transitive subgraph of C_n^k. The **cube** of a cycle is the 3rd power.

A drawing of a graph is **outerplane** (or **convex** or **circular**) if its vertices are placed along a circle (or the boundary of any convex region) and the edges are straight lines.

Two edges in an outerplane drawing **cross** if they intersect at a non-endpoint. Let (G,ω) denote the outerplane drawing of a graph G with cyclic order ω on $V(G)$.

A **book embedding** [4] of a graph G is an outerplane drawing and an edge-partition such that edges in the same part do not cross. The parts of the partition are the **pages** of the book embedding. The **book thickness** $bt(G)$ of G is the least number of pages in any book embedding while $bt(G,\omega)$ is the least number of pages for the outerplane drawing (G,ω).

A **proper edge-coloring** c of a graph G is a function $c: E(G) \to \{1,\ldots,r\}$ (the set of colors) such that adjacent edges get different colors. Let $\chi'(G)$ be the least number of colors in a proper edge-coloring. The remarkable theorem of Vizing [11, p 133] states that $\chi'(G) \in \{\Delta(G), 1+\Delta(G)\}$ for all graphs G.

A **matching book embedding** is a book embedding where the pages are matchings (no two edges are adjacent). The **matching book thickness** of a graph is the least number of pages in any matching book embedding; we write $mbt(G)$ or $mbt(G,\omega)$ as for book thickness. If c is the edge-coloring determined by the pages, then the matching book embedding is the triple (G,ω,c).

Clearly, for every graph G, we have $\Delta(G) \leq \chi'(G) \leq mbt(G)$. A matching book embedding (G,ω,c) is **dispersable** if the number $|c|$ of colors equals $\Delta(G)$ and is **nearly dispersable** [15] if $|c| = 1+\Delta(G)$. A graph is dispersable if it has a dispersable embedding and is nearly dispersable if it is not dispersable and has a nearly dispersable embedding. If G is regular and dispersable, then it is bipartite [21]. The **sparseness** $s(G,\omega,c)$ of a nearly dispersable book embedding is the least number of edges on any page. The sparseness $s(G)$ of a nearly dispersable graph G is the minimum sparseness over all minimum-page matching book embeddings.

Lemma 1 *Let G be a regular nearly dispersable graph of order n. Then the sparseness of G is at least 1 if n is even and at least $\Delta/2$ if n is odd.*

Proof: For n even, this is in Overbay [21], while for n odd, each page has at least one uncovered vertex, so for any set of Δ pages, there is a set of $\geq \Delta$ distinct points which need to be covered by edges from the remaining page. □

An infinite sequence $\{(G_n,\omega_n,c_n)\}_{n\geq 1}$ of matching book embeddings, such that $|E(G_n)|$ is strictly increasing, is called **sparse** (with sparseness s) if there exists k such that, for all n, we have (i) $\Delta(G_n) = k$, (ii) $mbt(G_n,\omega_n) = k+1$, (iii) $c_n : E(G_n) \to [k+1]$ is onto, and (iv) $s(G_n,\omega_n,c_n) = s$. For each n, the page with s edges is the **sparse** or "exceptional" page [18]; cf. [8].

It remains for us to define the m-fold *polymerization* of a matching book embedding of a circulant to form a circulant with the same set of jump-lengths but m-fold more vertices with no increase in the number of pages.

For $n \geq 7$ and nonempty $S \subseteq \{1,2,\ldots,\lceil\frac{n-1}{2}\rceil\}$, let $(C(n,S),\nu_n,c)$, $\nu_n := (1,\ldots,n)$, be a matching book embedding. We call an edge $e = a_ia_k \in E := E(C(n,S))$ **long** if $d_C(a_i,a_k) < |k-i|$ and **short** if $d_C(a_i,a_k) = |k-i|$, where $d_C(u,w)$ denotes the C_n-distance between two vertices u and w, where C_n is the graph induced by the cyclic vertex order. The sets E_Λ and E_Σ of long and short edges form a nontrivial partition of E. If $1 \in S$, then the edges a_ia_{i+1} are short for $i = 1,\ldots,n-1$ but the edge a_1a_n is long. If $n = 8$ and $3 \in S$, then $a_ia_{i+3} \in E_s$ for $1 \leq i \leq 5$, while a_6a_1, a_7a_2, and a_8a_3 are long.

Lemma 2 *Let $m \geq 2, n \geq 7$. If $S \subseteq \{1, 2, \ldots, \lceil \frac{n-1}{2} \rceil\}$, then*

$$mbt(C(nm, S), \nu_{nm}) \leq mbt(C(n, S), \nu_n). \tag{1}$$

Proof: Place m copies of the vertex set of $C(n, S)$ from left to right, where, for the j-th copy, the vertices $a_1^j, \ldots, a_n^j$ are placed from left to right. Thus,

$$(a_1^1, \ldots, a_n^1, a_1^2, \ldots, a_n^2, a_1^3, \ldots, a_n^m)$$

is a list of the nm vertices in the m copies of $C(n, S)$. Put in all the short edges for all the copies. For $j = 1, \ldots, m-1$, each long edge $a_i^j a_k^j$ with $k > i$ is replaced by $a_k^j a_i^{j+1}$ and each long edge $a_i^m a_k^m$ is replaced by $a_k^m a_i^1$ if $k > i$. One obtains $C(nm, S)$ with vertex order ν_{nm}. Use the same coloring c for the edges in $C(nm, S)$ as in the matching book embedding for $C(n, S)$. Long and short edges cross in the j-th copy if and only if their images under the edge-rearrangement cross correspondingly. Hence, c is a page assignment. $\square$

This process defines the m-fold **strict polymerization** of the circulant and of its matching book embedding. Note that equality can fail to hold in (1) - e.g., for an even polymerization of an odd cycle. If $(C(n, S), \nu_n, c)$ is dispersable, then so is $C(nm, S), \nu_{nm}, c)$, while if $(C(n, S), \nu_n, c)$ is nearly dispersable, then $C(nm, S), \nu_{nm}, c)$ is either dispersable or nearly dispersable. If both are nearly dispersable, then $s(C(nm, S), \nu_{nm}, c) \leq m \cdot s(C(n, S), \nu_n, c)$.

3 The bipartite case of $C(n, \{1, 3\})$

In this section we show that the circulants $C(n, \{1, 3\})$, $n \geq 6$ even, are dispersable. When $n = 6$, the corresponding circulant is $K_{3,3}$, and this graph, as with *all* complete regular bipartite graphs $K_{a,a}$, is dispersable [4], [21, p. 88].

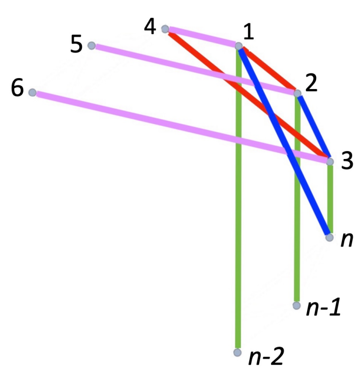

Figure 1: Common four-coloring c of the twist

Theorem 1 *Let $n \geq 8$, n even, and put $\tau_n := (1, 2, 3, n, n-1, n-2, \ldots, 6, 5, 4)$. Then $(C(n, \{1, 3\}), \tau_n, c_n)$ is a dispersable book embedding* (c_n defined below).

Proof: Let c be the 4-coloring of the 10 edges in the "twist" shown in Fig. 1:

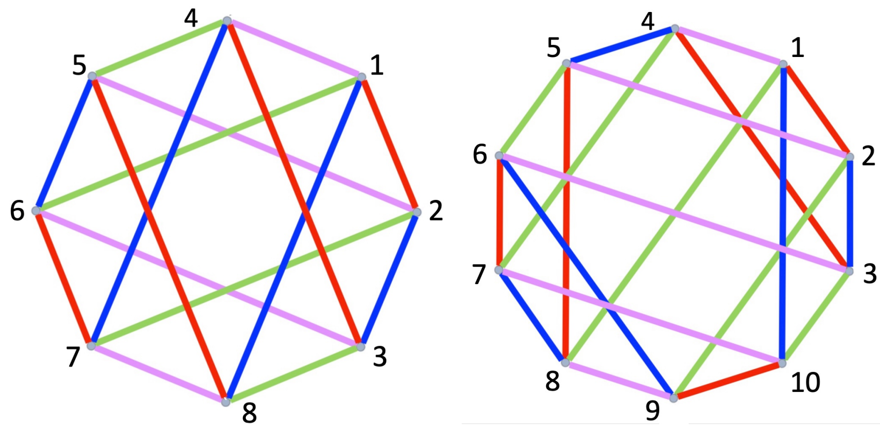

Figure 2: Dispersable embeddings of $C(8, \{1, 3\})$ and $C(10, \{1, 3\})$

- Red: 1−2, 3−4 (i.e., color of edge 1−2 is red, etc.),
- Blue: 2−3, 1−n,
- Green: 3−n, 2−(n−1), 1−(n−2),
- Purple: 1−4, 2−5, 3−6.

Case 1: $n = 4k$, $k \geq 2$. Assign the non-twist edges to four pages as follows:

- Blue: a−(a+3), (a+1)−(a+2), $a = 4t$, $t \in [k-1]$,
- Red: a−(a+3), (a+1)−(a+2), $a = 1 + 4t$, $t \in [k-1]$,
- Green: a−(a+3), (a+1)−(a+2), $a = 2 + 4t$, $t \in [k-2]$, and 4−5,
- Purple: a−(a+3), (a+1)−(a+2), $a = 3 + 4t$, $t \in [k-2]$, and (n−1)−(n).

All $8k$ edges of $C(n, \{1, 3\})$ appear, $2k$ in a page. In Fig. 2, left, $k = 2$, so $[k-2] = \emptyset$; hence, there is only one green edge not in the twist coloring.

On the Red and Blue pages, one has 2 edges on the common twist and $k-1$ edges of types a−($a+3$) and another $k-1$ edges of type (a+1)−(a+2). Similarly, on the Purple and Green pages, one has 3 edges on the common twist, an additional edge (4−5 or (n−1)−n) and $k-2$ each of types a−(a+3) and (a+1)−(a+2). These $8k$ edges are distinct and exhaust the edges of $C(n, \{1, 3\})$. For $n = 8$, $k = 2$ so $t = 1$ and on the Red page, $a = 5$. In Fig. 2, after 1−2 and 3−4, we also have 5−8 and 6−7 on the red page.

By definition, the edges on each of these pages are pairwise-disjoint, while pages are crossing-free since the edges in a color class can only be (i) non-crossing edges of the common twist, (ii) isolated edges on the outer cycle of the form 4−5 or (n−1)−n, (iii) nested edges of the form $\{a$−(a+3), (a+1)−(a+2)$\}$.

Case 2: Let $n = 4k + 2$, $k \geq 2$; put non-twist edges in four pages as follows:

- Green: a−(a+3), (a+1)−(a+2), $a = 4t$, $t \in [k-1]$,

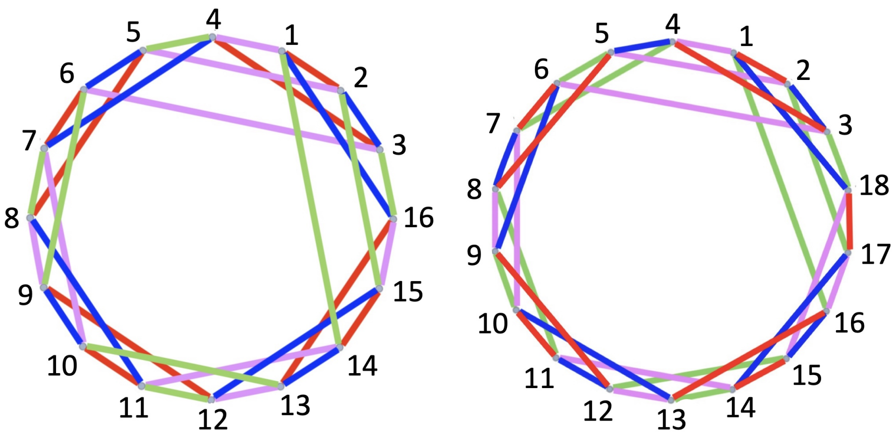

Figure 3: Dispersable embeddings of $C(16, \{1, 3\})$ and $C(18, \{1, 3\})$

- Red: a–$(a$+3), $(a$+1)–$(a$+2), $a = 1 + 4t$, $t \in [k-1]$, and n–$(n$–1),
- Blue: a–$(a$+3), $(a$+1)–$(a$+2), $a = 2 + 4t$, $t \in [k-1]$, and 4–5,
- Purple: a–$(a$+3), $(a$+1)–$(a$+2), where $a = 3 + 4t$, $t \in [k-1]$.

See Fig. 2, right. Fig. 3 shows both schemes for $k = 4$. □

4 The nonbipartite case of $C(n, \{1, 3\})$

When n is odd, $C(n, \{1, 3\})$ is not bipartite. But it is nearly dispersable.

Theorem 2 *Let $n \geq 7$ be odd. Then $C(n, \{1, 3\})$ is nearly dispersable.*

Proof: Case 1: $n = 4k + 1$. Let ν_n be the natural ordering $(1, 2, 3, \ldots, n)$ around the circle. We show that $mbt(C(n, \{1, 3\}), \nu_n) = 5$, which is minimum for a nonbipartite 4-regular graph. The $2n = 8k + 2$ edges are of the form u–$(u$+3) or u–$(u$+1), $u \in [n]$, with addition modulo n.

Assign each of the four colors red, purple, green, and blue to $2k$ edges and the fifth color, black, to the two remaining edges as follows:

- Red: a–$(a$+3), $(a$+1)–$(a$+2), $a = 1 + 4t$, $t \in [k] - 1$,
- Purple: a–$(a$+3), $(a$+1)–$(a$+2), $a = 2 + 4t$, $t \in [k] - 1$,
- Green: a–$(a$+3), $(a$+1)–$(a$+2), $a = 3 + 4t$, $t \in [k] - 1$,
- Blue: a–$(a$+3), $(a$+1)–$(a$+2), $a = 4 + 4t$, $t \in [k] - 1$,
- Black: 1–2, 3–n (sparseness is 2; as $\Delta = 4$, this is the minimum).

Since the edge pairs of the form $\{a$–$(a$+3), $(a$+1)–$(a$+2)$\}$ are nested for each value of a, which increases in increments of four for a fixed color, it is clear that no two edges of the same color cross. As a takes on all values from 1 to $4k$, these pairs of nested edges cover $8k$ distinct edges of the graph. In the case where $a = 4k + 1 = n$, the edge pair $\{a{-}(a{+}3), (a{+}1){-}(a{+}2)\}$ is $\{n{-}3, 1{-}2\}$, reducing mod n; these two edges fit on the fifth page without crossing.

This process, illustrated with $k = 3$ for the graph $C(13, \{1, 3\})$ in Fig. 4, may be viewed as taking the red coloring and rotating it three more times, switching colors for each rotation. Now add a fifth color for the last two edges.

One may also describe this family of colorings, using two Hamiltonian paths, alternatingly colored Red/Purple and Blue/Green. The Red/Purple path is $1, 4, 3, 2, 5, 8, 7, 6, 9, 12, 11, 10, 13$. See Fig. 4, left. As the reader will observe, there is a simple algorithm for the numbering: up 3, down 1, down 1, u3, u3, d1, d1, u3, u3, d1, d1, u3, and the resulting values mod 4 are periodic: $1, 0, 3, 2, \ldots$. The Blue/Green path is similar. The Hamiltonian paths each have $n - 1$ edges; the two missing edges in black are the sparse page.

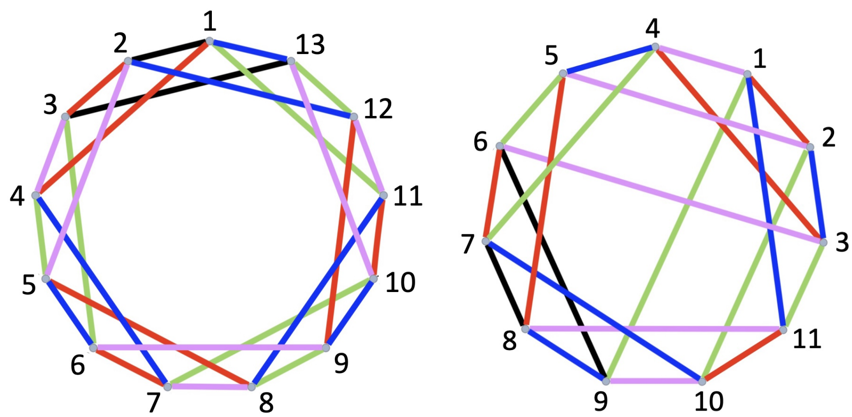

Figure 4: Nearly dispersable embeddings of $C(13, \{1, 3\})$ and $C(11, \{1, 3\})$

Case 2: $n = 4k + 3$. We show that $mbt(C(n, \{1, 3\}), \tau_n) = 5$, where τ_n is as in Theorem 1 above. Let c be the same edge-coloring for the 10 twist edges given in the proof of Theorem 1. If $k = 1$ (so $n = 7$), color the non-twist edges $6{-}7$ red, $4{-}5$ blue, and $4{-}7$ and $5{-}6$ black to get a 5-page matching book embedding. If $k \geq 2$, then we have a general coloring scheme

- Green: $a{-}(a{+}3)$, $(a{+}1){-}(a{+}2)$, $a = 4t$, $t \in [k - 1]$,
- Red: $a{-}(a{+}3)$, $(a{+}1){-}(a{+}2)$, $a = 1 + 4t$, $t \in [k - 1]$, and $n{-}(n{-}1)$,
- Purple: $n{-}(n{-}3)$, $(n{-}1){-}(n{-}2)$ (Sparse; has 5 edges including twist),
- Black: $a{-}(a{+}3)$, $(a{+}1){-}(a{+}2)$, $a = 2 + 4t$, $t \in [k - 1]$,
- Blue: $a{-}(a{+}3)$, $(a{+}1){-}(a{+}2)$, $a = 3 + 4t$, $t \in [k - 1]$, and $4{-}5$.

Non-twist edges of the same color are either nested pairs or or join consecutive vertices in the layout, so they cannot cross. The red and blue pages each contain $2(k-1)+1 = 2k-1$ non-twist edges, the green and black pages each have $2(k - 1) = 2k - 2$ non-twist edges, and the purple page contains 2 non-twist edges. The 5-page matching book embedding covers all $10 + 2(2k - 1) + 2(2k - 2) + 2 = 8k + 6 = 2(4k + 3) = 2n$ distinct edges of the graph. Fig. 4 (right) illustrates this coloring scheme for $C(11, \{1, 3\})$ with $k = 2$. $\square$

Note that in the above theorem, when $n > 7$ and $n = 4k + 3$, the two purple common twist edges 1−4 and 2−5 could be assigned the color black. This would reduce the number of purple edges from five to three. As $n \equiv 3 \pmod 4$ increases, the sparse purple page stays at three edges. Hence, sparseness is 3, one more than the minimum.

5 $C(n, \{1, 2\})$ is nearly dispersable

Now we consider the circulant graphs $C(n, \{1, 2\})$ for all $n \geq 4$. In the case $n = 4$ the corresponding graph is K_4, which has $mbt(G) = \Delta(G) + 1 = 4$.

Theorem 3 *Let $n \geq 5$. Then $C(n, \{1, 2\})$ is nearly dispersable.*

Proof: We color this circulant in three cases using paths and cycles.

For $n \geq 5$ odd, draw the circulant using the odd-up, even-down cyclic order

$$\omega_n := (1, 3, \ldots, n, n-1, n-3, \ldots, 2).$$

Color the edges of the "zig-zag" (Hamiltonian) path $1, 2, 3, \ldots, n$ using the colors red, blue alternatingly for the edges $(k-1)-k$, where $k = 2, 3, \ldots, n$, along the path. Color alternatingly with purple and green, the "cross path"

$$3, 5, \ldots, n, 1, n-1, n-3, \ldots, 2.$$

This leaves two edges 1−3 and $2-n$ each of which can be colored black. Thus, $mbt(G, \omega_n) = 5$ and the sparseness is minimum. See Fig. 5, left.

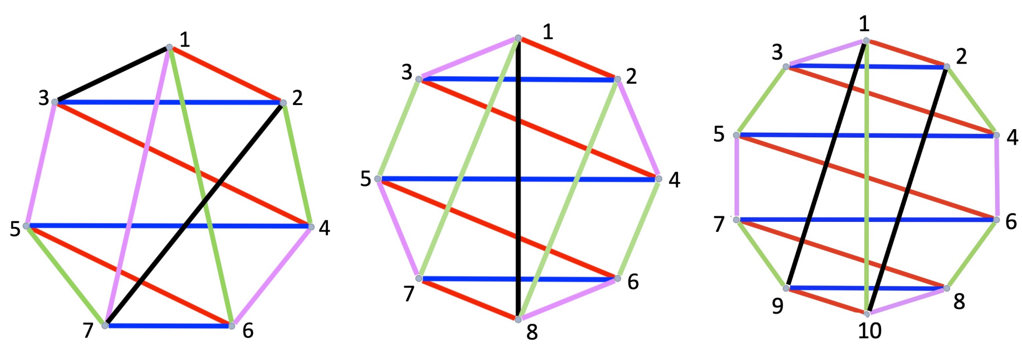

Figure 5: $C(7, \{1, 2\})$, $C(8, \{1, 2\})$ $C(10, \{1, 2\})$ are nearly dispersable.

For $n \geq 6$ even, again use the odd-up, even-down cyclic order

$$\omega_n := (1, 3, \ldots, n-1, n, n-2, \ldots, 2).$$

As above, color the edges of the zig-zag path $1, 2, 3, \ldots, n$ using the colors red, blue alternatingly. If $n \equiv 0 \pmod 4$, color the two disjoint non-self-crossing cycles $n-1, n-3, \ldots, 1, n-1$ and $2, 4, \ldots, n, 2$ (both of length $n/2$) alternating purple and green. This accounts for $2n - 1$ edges and the remaining edge $1-n$ is colored black. The sparseness is 1 which is the minimum possible for an even order, nearly dispersable graph. See Fig. 5, center.

If $n \equiv 2 \pmod 4$, edge-disjoint from the zig-zag path, there is another Hamiltonian path $n{-}1, n{-}3, \ldots, 1, n, n{-}2, \ldots, 2$ and we color it alternatingly green and purple. The two remaining edges $1{-}(n{-}1)$ and $2{-}n$ are parallel and colored black, see Fig. 5, right; they constitute the sparse page. □

6 The case $C(n, \{2, 3\})$

Let ω_n be the odd-up, even-down vertex-order from the proof of Theorem 3.

Theorem 4 *If $n \geq 8$ is* even*, then $(C(n, \{2, 3\}), \omega_n, c_n)$ is nearly dispersable, where c_n is the coloring given below.*

Proof: Put $n = 2k$; consider the Hamiltonian path in $C(n, \{2, 3\})$ given by

$$(n{-}3, n{-}5, \ldots, 1, n{-}1, 2, n, n{-}2, \ldots, 4)$$

which is noncrossing w.r.t. the layout ω_n; we color it alternatingly red and black. This colors $2k-1$ edges. Color with purple the pairwise-disjoint edges $2{-}5$, $4{-}7$, …, $(n{-}4){-}(n{-}1)$, which accounts for $k-2$ edges. Color with blue the pairwise-disjoint edges $1{-}4$, $3{-}6$, …, $(n{-}3){-}n$, accounting for another $k-1$ edges. In the sparse green page, there are four edges: $2{-}4$, $(n{-}3){-}(n{-}1)$, $3{-}n$, and $1{-}(n{-}2)$. Hence, all $4k$ edges are used. As same-color edges do not cross or share an endpoint, ω_n is nearly dispersable. See Fig. 6, left. □

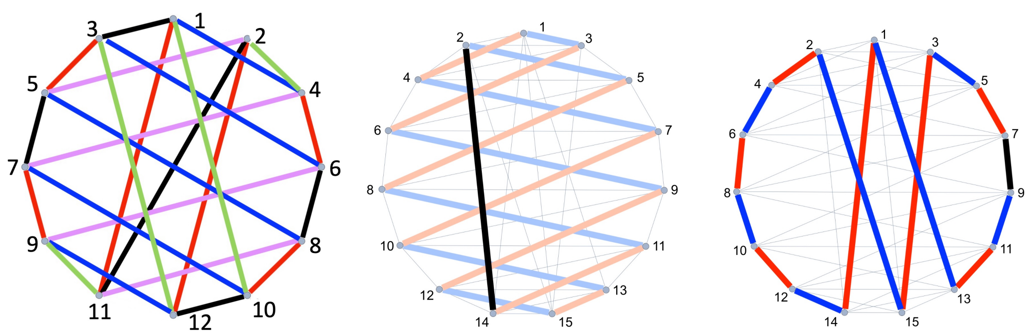

Figure 6: $C(12, \{2, 3\})$ on left, pages of $C(15, \{2, 3\})$ in middle and right.

That $C(6, \{2, 3\})$ is nearly dispersable is left to the reader.

Theorem 5 *For $n \geq 7$* odd*, $C(n, \{2, 3\})$ is nearly dispersable w.r.t. ω_n.*

Proof: Let $n = 2r + 1 \geq 7$. We decompose the edge set into two edge-disjoint Hamiltonian cycles. These odd-length cycles are 2-colored except for one black edge, placed on the sparse page. The first of these two cycles, see Fig. 6-middle, is colored orange, aqua, and black according to the following scheme:

- Orange: $a{-}(a{+}3)$, $a = 1 + 2t$, $t \in [r-1] - 1$ and $n{-}(n{-}2)$,
- Aqua: $a{-}(a{+}3)$, $a = 2t$, $t \in [r-1]$ and $1{-}3$,

- Black: $2-(n-1)$.

The second of these cycles, illustrated in Fig. 6-right, is colored red, blue, and black according to the following scheme:

- Red: $1-(n-1)$, $3-n$, and $r-2$ additional edges of the form $a-(a+2)$ on the outer cycle (alternating with the blue edges),
- Blue: $2-n$, $1-(n-2)$, and $r-2$ additional edges of the form $a-(a+2)$ on the outer cycle (alternating with the red edges),
- Black: $r-(r+2)$.

The orange and aqua pages each consist of a total of r non-crossing parallel edges. The red and blue pages also each contain r non-crossing edges with two parallel edges on each page through the center and the remaining $r-2$ edges on the outer boundary. The remaining two black edges, which is the minimum possible number, clearly do not intersect on the sparse page since the $r-(r+2)$ edge lies on the outer cycle and does not share an endpoint with $2-(n-1)$. Hence, all $4r+2=2n$ edges are accounted for. $\square$

Theorem 6 *For* $n \geq 7$ odd*,* $C(n, \{1, 2, 3\})$ *is nearly dispersable w.r.t.* ω_n*.*

Proof:

Let $n = 2r+1 \geq 7$. Use the identical layout and coloring scheme as in Theorem 5. This will cover all distance 2 and distance 3 edges. Now observe that a purple-green non-crossing Hamiltonian path $1, 2, 3, \ldots, n$ can be added to cover all of the distance-1 edges, with the exception of edge $1-n$. This last edge can be placed on the black (sparse) page and does not intersect either $r-(r+2)$ or $2-(n-1)$. We note that this new cycle contributes r purple edges, r green edges, and 1 black edge as shown in Fig. 7 left. Combining this with $C(n, \{2, 3\})$, we have accounted for all $6r+3=3n$ edges of $C(n, \{1, 2, 3\})$ and have achieved an optimal sparseness of $3 = \Delta/2$. See Fig. 7 right for the combined nearly-dispersable coloring of $C(n, \{1, 2, 3\})$ for n odd. $\square$

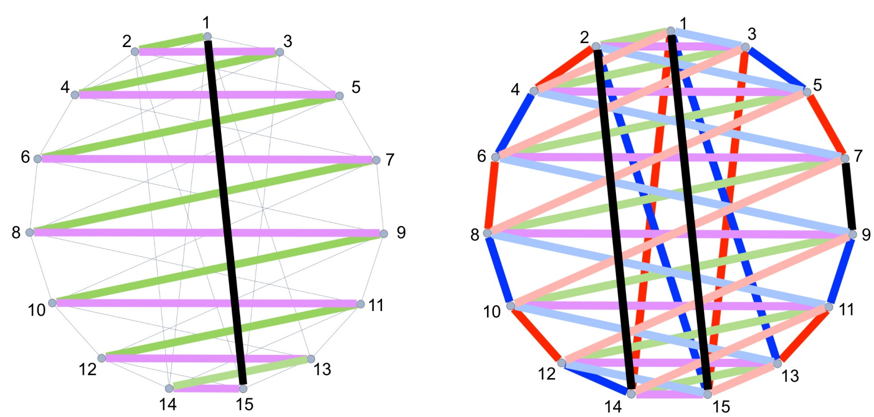

Figure 7: Length-1 edges (left) and 7-coloring of $C(15, \{1, 2, 3\})$ (right).

It is natural to ask if the above scheme for adding distance-1 edges allows *even values* of n. This almost works, with the exception of the black edge $1{-}n$, which intersects edges on the sparse page in the above layout for even values of n, so for even n, $mbt(C(n, \{1, 2, 3\}) - e) = 7$, where $e = 1{-}n$.

7 Larger degree and jump-lengths

We now give minimum layouts of $C(n, \{1, 2, 3\})$ for some even values of n and for a variety of circulants with $\Delta > 3$, using polymerization and periodicity.

Theorem 7 *With $\nu_n = (1, 2, \ldots, n)$ and $r = 2k + 1 \geq 5$, suppose $r|n$. Then*

$$mbt(C(n, \{1, 2, \ldots, k\}), \nu_n) = 2k + 1.$$

Proof: Use Lemma 2 on the embedding (K_r, ν_r, c) in [21]; see Fig 8 and 9. □

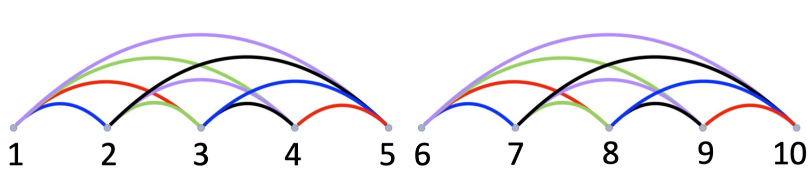

Figure 8: Two copies of K_5 .

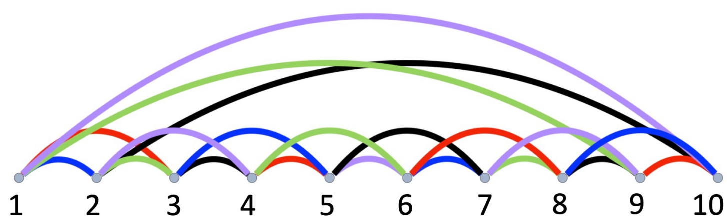

Figure 9: $C(10, \{1, 2\})$ obtained by polymerization.

For $r = 7$, and for all $m \geq 1$, one has the consequence (new for m even):

$$mbt(C(7m, \{1, 2, 3\})) = 7. \tag{2}$$

Similarly, $mbt(C(9m, \{1, 2, 3, 4\})) = 9$, $mbt(C(11m, \{1, 2, 3, 4, 5\})) = 11$, etc.

Note that, for k a positive integer, $C(n, [k]) \cong K_n$ for $n \in \{2k, 2k + 1\}$, but the isomorphism determines *which* edge-set E_j of K_n corresponds to the length-j jumps in the circulant for $j = 1, \ldots, k$. With $n = 2k$ or $2k + 1$, the length-k jumps are longest in the circulant and they induce a 1-factor or a spanning cycle according to whether n is even or odd.

Corresponding to the $n = 2k$ case, for $k \geq 3$, the **cocktail party graph** $O_k := K_{2k} - kK_2$ is the complement of a 1-factor. It is also the 1-skeleton of the n-dimensional **octahedron** and is regular with $\Delta O_k = 2k - 2$. As it contains triangles, the octahedron is at best nearly dispersable.

We show that it does have a nearly dispersable embedding, with $2k-1$ pages, but the embedding does not use the standard vertex ordering and so we don't have a direct way to polymerize it. However, the natural vertex order gives a matching book embedding with one additional page and this *can* be polymerized. For the $n = 2k + 1$ case, the same things can be done with $\overline{C_n}$, the complement of C_n.

The **folded order** ϕ from [18] will be needed for $n \in \{2k, 2k+1\}$.

$$\phi_n := (1, 2, \ldots, k, n, n-1, \ldots, k+1).$$

Theorem 8 *For $k \geq 3$, O_k and $\overline{C}_{2k+1}$ are nearly dispersable with*

$$mbt(O_k, \phi_{2k}) = 2k - 1 \; = \; mbt(\overline{C}_{2k+1}, \phi_{2k+1}). \tag{3}$$

Proof: For $n \geq 6$, even or odd, the complete graph K_n is nearly dispersable using the natural vertex order with pages being the 1-factors produced by maximal families of parallel edges given in [21, p 87]; see Fig. 10 and Fig. 12.

By our remark above about how the edges can correspond to various jump-lengths in the isomorphic circulant graph, we note that for $n = 2k$ and the folded order ϕ_{2k}, the set of length-k edges, E_k, are a parallel matching, one of the $2k$ pages in the nearly dispersable matching book embedding of K_{2k}, and the removal of E_k leaves a $2k-1$-page layout of O_k. See right side of Fig. 11.

For $n = 2k+1$, the length-k edges in the circulant constitute a Hamiltonian cycle Z. The folded order takes this $2k+1$-cycle into 2 of the $2k+1$ pages of the standard matching book embedding of K_{2k+1} given in [21], with one extra edge. Deleting Z gives a $2k-1$-page layout of $\overline{C}_{2k+1}$ (Fig. 13, right). ☐

With one page above the minimum, we have *polymerizable* embeddings,

$$mbt(O_k, \nu_{2k}) = 2k = mbt(\overline{C}_{2k+1}, \nu_{2k+1}).$$

Under the natural order, the edges of K_{2k} which are length-k edges in $C(2k, [k])$ pass through the center of the circle. Hence, removing them only decreases by 1 the number of edges in each of the $2k$ pages. See left side of Fig. 11. Similarly, with the layout ν_{2k+1}, the length-k edges form a mandala-like figure (Fig 13, left) which can be deleted from the matching book embedding.

Thus, if n is a multiple of $2k$ or of $2k+1$ for $k \geq 3$, then by Lemma 2,

$$mbt(C(n, \{1, \ldots, k-1\}), \nu_n) \leq 2k. \tag{4}$$

The next theorem allows multiples of 12.

Theorem 9 *If $n = 12m$, $m \geq 1$, then $C(n, \{1, 2, 3\})$ is nearly dispersable.*

Proof: An explicit coloring of the edges with respect to natural order ν_n using $7 = 1 + \Delta$ colors is given as follows: Periodically, 4-color the edges of length 1, and for every edge of length 3, use the same color as the unique edge of length 1 with which it is nested, thus 4-coloring all edges of length 1 or 3. Three new colors (periodically) suffice for the remaining edges. See Fig. 14. ☐

For the bipartite case, if $k \geq 1$, by Lemma 2, analogous to (4), we have:

$$\text{If } 4k|n, \text{ then } mbt(C(n, \{1, 3, 5, \ldots, 2k-1\}), \nu_n) = 2k; \tag{5}$$

$$\text{If } (4k+2)|n, \text{ then } mbt(C(n, \{1, 3, 5, \ldots, 2k-1\}), \nu_n) \leq 2k+1. \tag{6}$$

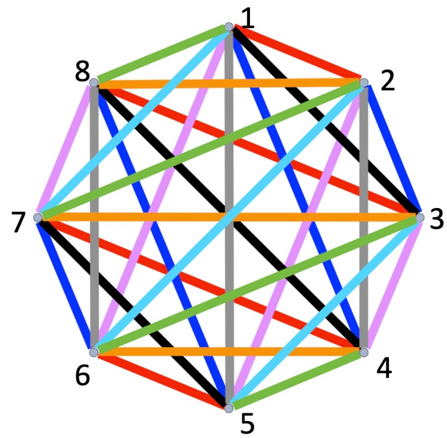

Figure 10: K_8.

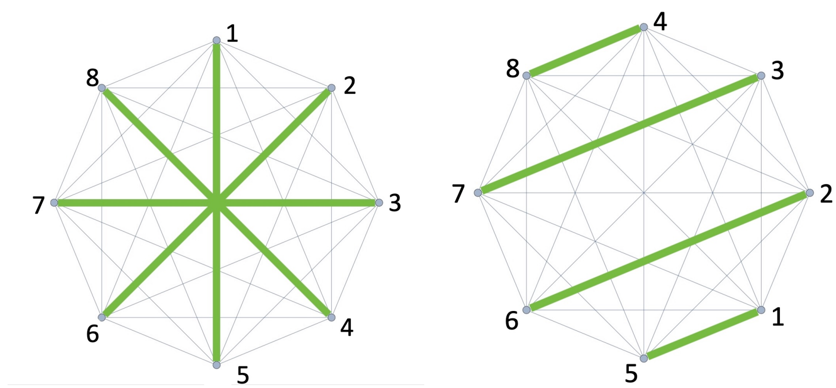

Figure 11: A perfect matching in K_8.

8 Discussion

Recently, Yu, Shao and Li [27] have shown dispersability (or near dispersability) for circulants of degree 3 and degree 4 with *all* jump-lengths according to whether or not they are bipartite. This extends our results for these degrees. Our methods supply different solutions to the problem of finding optimal page-number matching book embeddings for such circulants.

We have also considered higher degree circulants and have analyzed some of the structural features of matching book embeddings of regular graphs.

For $C(n, \{1, 2, 3\})$, we show near dispersability when n is odd and when n is even and divisible by 7 or 12. We also show near dispersability for the circulants resulting by deleting a maximum matching from an even-order K_n and a spanning cycle from an odd-order K_n. The analogous results holds in the bipartite case.

Results are constructive, not just existential, and so will remain useful even if the full conjecture on vertex-transitive graphs is proved (or disproved).

Some condition on the graph is needed as a regular graph can have an *arbitrarily large* value

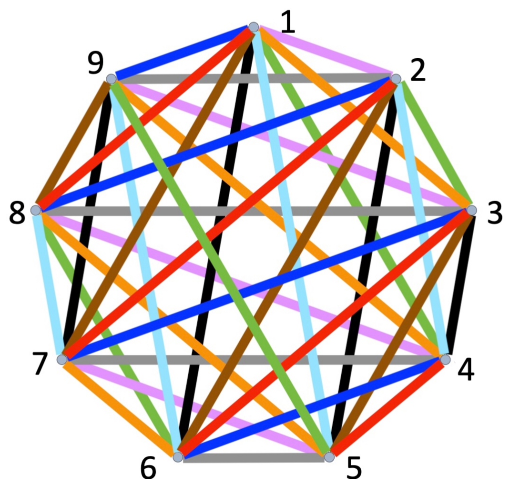

Figure 12: K_9.

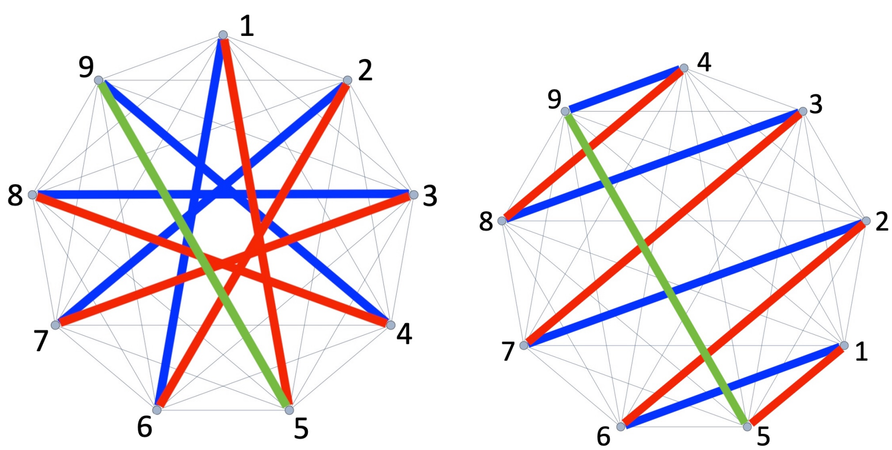

Figure 13: A Hamiltonian cycle in K_9.

for the ratio mbt/Δ according to Alam et al. [2], which uses a counting argument of McKay [20] to prove that, for any fixed $\Delta \geq 3$, there exist Δ-regular bipartite graphs G_n with $mbt(G_n) \to \infty$ as $n \to \infty$.

But vertex-transitive graphs are rather special. Du, Kutnar & Marušič [7] showed that the Lovasz conjecture (*Every vertex transitive graph contains a Hamiltonian cycle, with five exceptional cases*) is correct when the order is a product of two primes and the graph satisfies additional conditions involving the action of a group. Also, Diestel [6, p 52; Ex 12] notes that every connected, even-order, vertex-transitive graph has a 1-factor.

In many cases, our proof of near dispersability for a family of matching book embeddings uses a sparse page with the minimum number of edges. Indeed, the lower bound of 2 is achieved in the proof of Theorem 2 for $C(n, \{1,3\})$ when $n \equiv 1 \pmod 4$ while we get sparseness $s \leq 3$ for $n \equiv 3 \pmod 4$. The proof of Theorem 3 shows $C(n, \{1,2\}) = 2$ for n odd, while for $n \equiv 0$ (mod

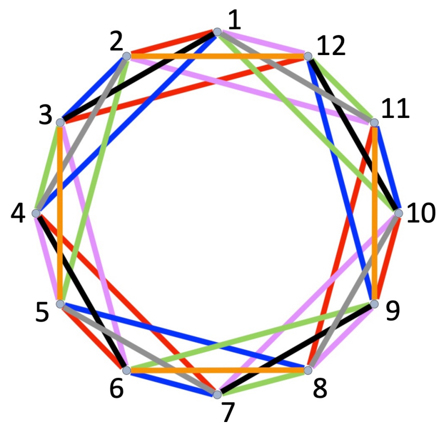

Figure 14: $C(12, \{1, 2, 3\})$.

4), sparseness = 1 is achieved but for $n \equiv 2 \pmod 4$, we only get sparseness ≤ 2. The proof of Theorem 4 shows $s(C(n, \{2, 3\})) \leq 4$ if n is even. Sparseness has minimum value (2 and 3, resp.) for Theorems 5 and 6 when n odd and for degree 6. Our matching book embeddings for Theorems 7, 8 and 9, in contrast, are quite symmetric and so the opposite of sparse embeddings.

For a nearly dispersable embedding, sparseness allows the deletion of a small number of edges to eliminate an entire page, while symmetry might be preferred for an "online" problem where the graph being embedded is evolving.

Matching book thickness for regular graphs has a clear lower bound, so a coloring which achieves the minimum is detectable. We think that finding the matching book thickness of various graphs could be a good target for genetic algorithms, neural networks, or artificial intelligence. See, for example, [3]. Application of machine learning techniques to matching book thickness might improve computational theory and practice; see, e.g., [13]. Indeed, one has an endless supply of vertex-transitive graphs on which to test procedures.

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