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Lower bounds for Ramsey numbers for complete bipartite and 3-uniform tripartite subgraphs

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Abstract

Let $R(K_{a,b}, K_{c,d})$ be the minimum number n so that any n-vertex simple undirected graph G contains a $K_{a,b}$ or its complement G' contains a $K_{c,d}$. We demonstrate constructions showing that $R(K_{2,b}, K_{2,d}) > b + d + 1$ for $d \geq b \geq 2$. We establish lower bounds for $R(K_{a,b}, K_{a,b})$ and $R(K_{a,b}, K_{c,d})$ using probabilistic methods. We define $R'(a, b, c)$ to be the minimum number n such that any n-vertex 3-uniform hypergraph $G(V, E)$, or its complement $G'(V, E^c)$ contains a $K_{a,b,c}$. Here, $K_{a,b,c}$ is defined as the complete tripartite 3-uniform hypergraph with vertex set $A \cup B \cup C$, where the A, B and C have a, b and c vertices respectively, and $K_{a,b,c}$ has abc 3-uniform hyperedges $\{u, v, w\}$, $u \in A$, $v \in B$ and $w \in C$. We derive lower bounds for $R'(a, b, c)$ using probabilistic methods. We show that $R'(1, 1, b) \leq 2b + 1$. We have also generated examples to show that $R'(1, 1, 3) \geq 6$ and $R'(1, 1, 4) \geq 7$.

Keywords: Ramsey number, bipartite graph, local lemma, probabilistic method, r-uniform hypergraph.

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1 Introduction

Let $R(G_1, G_2)$ denote the smallest integer such that for every undirected graph G with $R(G_1, G_2)$ or more vertices, either (i) G contains G_1 as subgraph, or (ii) the complement graph G' of G contains G_2 as subgraph. In particular $R(K_{a,b}, K_{a,b})$ is the smallest integer n such that any n-vertex simple undirected graph G or its complement G' must contain the complete bipartite graph $K_{a,b}$. Equivalently, $R(K_{a,b}, K_{a,b})$ is the smallest integer n of vertices such that any bicoloring of the edges of the n-vertex complete undirected graph K_n would contain a monochromatic $K_{a,b}$. The significance of such a number is that it gives us the minimum number of vertices needed in a graph so that two mutually disjoint subsets of vertices with cardinalities a and b can be guaranteed to have the complete bipartite connectivity property as mentioned. In the analysis of social networks it may be worthwhile knowing whether all persons in some subset of a persons share b friends, or none of the a persons of some other subset share friendship with some set of b persons. This can also be helpful in the analysis of dependencies, where there are many entities in one partite set, which are all dependent on entities in the other partite sets; we need to achieve consistencies that either all dependencies exist between a pair of partite sets, or none of the dependencies exist between possibly another pair of partite sets.

1.1 Existing results

From the definition of $R(K_{a,b}, K_{a,b})$, it is clear that $R(K_{1,1}, K_{1,1})$=2 and $R(K_{1,2}, K_{1,2})$=3. To see that $R(K_{1,3}, K_{1,3}) \geq 6$, observe that we need at least 4 vertices and neither a 4-cycle nor its complement has a $K_{1,3}$. Further, observe that neither a 5-cycle in K_5, nor its complement (also a 5-cycle) has a $K_{1,3}$. The numbers $R(K_{1,b}, K_{1,b})$ are however known exactly, and are given by Burr and Roberts [3] as $R(K_{1,b}, K_{1,b}) = 2b - 1$ for even b, and $2b$, otherwise. Chvátal and Harary [5] were the first to show that $R(C_4, C_4) = 6$, where C_4 is a cordless cycle of four vertices. As $K_{2,2}$ is identical to C_4, $R(K_{2,2}, K_{2,2}) = 6$. Note that $R(K_{2,3}, K_{2,3}) = 10$ [2], $R(K_{2,4}, K_{2,4}) = 14$ [7], and $R(K_{2,5}, K_{2,5}) = 18$ [7]. The values of $R(K_{2,2}, K_{2,n})$ are known to be exactly 20, 22, 25, 26, 30 and 32 for $n = 12, 13, 16, 17, 20$ and 21, respectively [12]. For integers n such that $12 \leq n \leq 16$, the values of $R(K_{2,n}, K_{2,n})$ are known to be exactly 46, 50, 54, 57 and 62, respectively [12]. Harary [8] proved that $R(K_{1,n}, K_{1,m}) = n + m - x$, where $x = 1$ if both n and m are even and $x = 0$ otherwise. These Ramsey numbers are different from the usual Ramsey numbers $R(K_a, K_b)$, where $R(K_a, K_b)$ is the smallest integer n such that any undirected graph G with n or more vertices contains either a K_a or an independent set of size b. We know that $R(K_3, K_3) = 6$, $R(K_3, K_4) = 9$, $R(K_3, K_8) = 28$, $R(K_3, K_9) = 36$, $R(K_4, K_4) = 18$, and $R(K_4, K_5) = 25$ (see [12, 14, 13]).

1.2 Our contribution

We derive lower bounds for (i) the *unbalanced diagonal case* for $R(K_{a,b}, K_{a,b})$ and (ii) the *unbalanced off-diagonal case* for $R(K_{a,b}, K_{c,d})$. In Section 2 we also establish a lower bound of $2b + 1$ for $R(K_{2,b}, K_{2,b})$ for all $b \geq 2$. We provide an explicit construction and use combinatorial arguments. Note that Lortz and Mengersen [9] conjectured that $R(K_{2,b}, K_{2,b}) \geq 4b - 3$, for all $b \geq 2$. Exoo et al. [7] proved that $R(K_{2,b}, K_{2,b}) \leq 4b - 2$ for all $b \geq 2$, where the equality holds if and only if a *strongly regular* $(4b - 3, 2b - 2, b - 2, b - 1)$-graph exists. (A k-regular graph G with n vertices is called *strongly regular* graph (n, k, p, q), if every adjacent pair of vertices shares exactly p neighbours and every non-adjacent pair of vertices shares exactly q neighbours.) In Sections 2 and 3, we consider Ramsey numbers $R(K_{a,b}, K_{a,b})$ and $R(K_{a,b}, K_{c,d})$ respectively, where a, b, c and d and integers, establishing lower bounds using probabilistic methods. In Section 3, we also demonstrate a construction showing that $R(K_{2,b}, K_{2,d}) > b + d + 1$ for $d \geq b \geq 2$. In Section 4 we extend similar methods for 3-uniform tripartite hypergraphs, deriving lower bounds for the Ramsey numbers $R'(a, b, c)$; we are unaware of any literature concerning such lower bounds for such hypergraphs. Here, $R'(a, b, c)$ is the minimum number n such that any n-vertex 3-uniform hypergraph $G(V, E)$, or its complement $G'(V, E^c)$ contains a $K_{a,b,c}$. Here, $K_{a,b,c}$ is defined as the complete tripartite 3-uniform hypergraph with vertex set $A \cup B \cup C$, where the A, B and C have a, b and c vertices respectively, and $K_{a,b,c}$ has abc 3-uniform hyperedges $\{u, v, w\}$, $u \in A$, $v \in B$ and $w \in C$. In Section 4, we also show that $R'(1, 1, b) \leq 2b + 1$. Further, we present our generated examples to show that $R'(1, 1, 3) \geq 6$ and $R'(1, 1, 4) \geq 7$. In Section 5 we conclude with a few remarks.

2 The unbalanced diagonal case : $R(K_{a,b}, K_{a,b})$

$R(K_{a,b}, K_{a,b})$ is the minimum number n of vertices such that any bicoloring of the edges of the n-vertex complete undirected graph K_n would contain a monochromatic $K_{a,b}$. The following lower bound proof for $R(K_{2,b}, K_{2,b})$ involves an explicit construction. We are not aware of better lower bounds for $R(K_{2,b}, K_{2,b})$ in the literature.

2.1 A constructive lower bound for $R(K_{2,b}, K_{2,b})$

Theorem 1 $R(K_{2,b}, K_{2,b}) > 2b + 1$, *for all integers* $b \geq 2$.

Proof: For $b \geq 2$, there always exists a graph G with $2b + 1$ vertices, such that neither G nor its complement G' has a $K_{2,b}$. The entire construction is illustrated in Figure 1. Let the vertices be labelled v_1, v_2, ..., v_{2b+1} for $b = 4$. Connect v_{2b+1} to each of the other $2b$ vertices. Let B_1 be the set of vertices v_1, v_2, ..., v_b and B_2 be the set of vertices v_{b+1}, v_{b+2}, ..., v_{2b}. We wish to connect every vertex in B_1 to at most $b - 1$ vertices in B_2. There are $\binom{b}{b-1} = b$ such

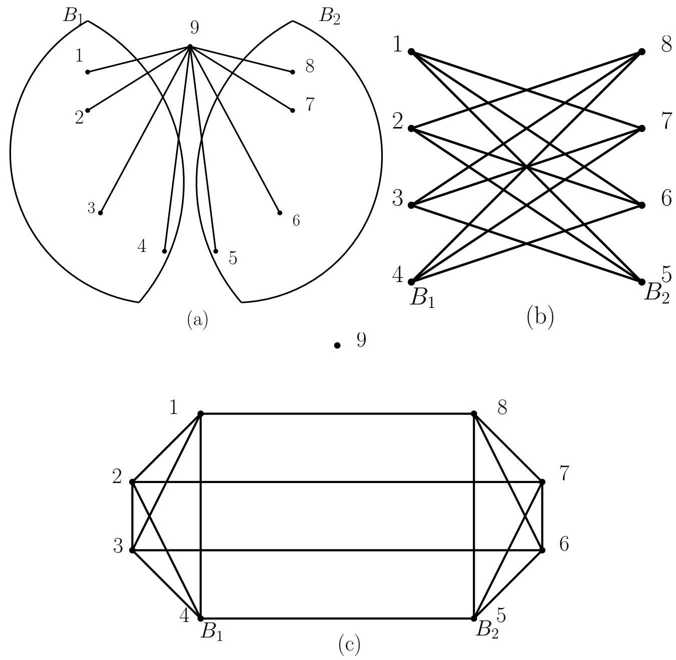

Figure 1: (a) Edges of G connecting v_{2b+1}, (b) Edges of G between vertices of B_1 and B_2, (c) the complement graph G' of G, wherein B_1 and B_2 are a K_b each, and B_1 and B_2 have a perfect matching.

distinct subsets of B_2 of size $b-1$. Now each vertex v_i of B_1 is connected to $b-1$ vertices of B_2, leaving out only the vertex v_{2b+1-i} of B_2. We claim that the degree of every vertex except v_{2b+1} is b. Firstly, every vertex of B_1 is connected to $b-1$ vertices of B_2, and the single vertex v_{2b+1}. Secondly, every vertex of B_2 (i) is connected to v_{2b+1}, and (ii) also present in exactly $b-1$ separate groups, where each group is connected to exactly one vertex of B_1. So, every vertex v_j of B_2 is connected to every vertex of B_1 except the vertex v_{2b+1-j} (in B_1). So, every vertex of $B_1 \cup B_2$ has degree b. However, no two vertices in $B_1 \cup B_2$ have all b identical neighbours. Therefore, G is $K_{2,b}$-free.

Now consider G'. Since v_{2b+1} is connected to every other vertex in G, it is isolated in G'. Since each vertex in G is connected to $b-1$ vertices other than v_{2b+1}, the number of neighbours for each vertex in G' is precisely $(2b-1)-(b-1) = b$, as illustrated in Figure 1. We show that for any two vertices in G', their neighbouring sets of b vertices in G' differ in at least one vertex. Observe that in G', B_1 and B_2 include complete graphs K_b, and the edges between B_1 and B_2 form a perfect matching. Consequently, the neighbouring sets of any two vertices differ by at least one vertex in G'. Since the number of common neighbours between any two vertices is no more than $b-1$, G' is also $K_{2,b}$-free.$\square$

2.2 Probabilistic lower bounds for $R(K_{a,b}, K_{a,b})$

In the first Section 2.2.1 we use the probabilistic method that Erdós applied to prove lower bounds on the original Ramsey numbers [6]. In the Section 2.2.2, we demonstrate improved lower bounds using the Loväsz local lemma.

2.2.1 Application of the probabilistic method

The best known lower bound on $R(K_{a,b}, K_{a,b})$ due to Chung and Graham [4] is

$$R(K_{a,b}, K_{a,b}) > \left(2\pi\sqrt{ab}\right)^{\left(\frac{1}{a+b}\right)} \left(\frac{a+b}{e^2}\right) 2^{\frac{ab-1}{a+b}} \tag{1}$$

Table 1: Lower bounds for $R(K_{a,b}, K_{a,b})$ from Inequality 1(left), Theorem 2 (middle) and Theorem 3 (right)

b / a	4	5	6	7	8	14	15	16
1	2,3,4	3,4,5	3,5,6	3,5,7	3,6,8	5,10,17	5,11,18	6,12,19
2	3,5,6	4,6,7	5,7,9	5,8,10	6,9,12	9,17,23	10,18,24	10, 19, 26
3	5,7,8	6,8,9	7,10,12	8,12,14	9,14,16	16,26,32	17,29,35	18,31,37
4	6,9,10	8,11,12	10,14,15	12,16,18	14,19,22	26,41,46	28,45,50	30,49,55
5		11,14,16	13,18,20	16,22,24	19,27,29	40,60,65	43,67,72	47,74,80
6			17,23,25	21,29,31	26,35,38	59,87,93	66,98,104	72,109,116
7				27,37,39	34,46,48	86,123,129	96,139,147	106,156,165
8					43,58,61	119,168,178	136,193,204	152,219,232
14						556,755,820	678,922,1005	817,1113,1219
15							836,1136,1246	1019,1385,1525
16								1254,1704,1886

We derive a tighter lower bound using the probabilistic method as follows.

Theorem 2 *$R(K_{a,b}, K_{a,b}) > \frac{\left(a^a b^b 2\pi\sqrt{ab}\right)^{\left(\frac{1}{a+b}\right)} 2^{\left(\frac{ab-1}{a+b}\right)}}{e}$, for natural numbers a and b.*

Proof: First we find the probability p of existence of a particular monochromatic $K_{a,b}$ and then sum that probability over all such possible distinct complete bipartite graphs to estimate an upper bound on the probability of existence of some monochromatic $K_{a,b}$. To get a lower bound on $R(K_{a,b}, K_{a,b})$, we choose the largest value of n, keeping the probability p strictly less than unity. This would ensure the existence of some graph G with n vertices such that both G and G' are free from any monochromatic $K_{a,b}$. Let n be the number of vertices of graph G. Then the total number of distinct $K_{a,b}$'s possible is $\binom{n}{a}\binom{n-a}{b}$. Each $K_{a,b}$ has exactly ab edges. Each edge can be either of two colors with equal probability. The probability that a particular $K_{a,b}$ will have all ab edges of a specific color is $\left(\frac{1}{2}\right)^{ab}$. So, the probability that a particular $K_{a,b}$ is monochromatic is $2\left(\frac{1}{2}\right)^{ab} = 2^{1-ab}$. The probability p that some $K_{a,b}$ is monochromatic is $\binom{n}{a}\binom{n-a}{b}2^{1-ab}$. Our objective is to choose as large n as possible with $p < 1$. So, choosing $n > \frac{(a^a b^b 2\pi\sqrt{ab})^{\left(\frac{1}{a+b}\right)} 2^{\left(\frac{ab-1}{a+b}\right)}}{e}$, for natural numbers a and b, and using Stirling's approximation (replacing $a!$ by $\sqrt{2\pi}\frac{a^{a+\frac{1}{2}}}{e^a}$ and $b!$ by $\sqrt{2\pi}\frac{b^{b+\frac{1}{2}}}{e^b}$), we get $p < 1$. This guarantees the existence of an n-vertex graph for which some edge bicoloring would not result in any monochromatic $K_{a,b}$. □

See Table 1 for the first two lower bounds for $R(K_{a,b}, K_{a,b})$ for each pair (a, b), due to Inequality 1 and Theorem 2, respectively. Taking the ratio of our lower bound in Inequality 2, and Chung and Graham's lower bound as in Inequality 1, we get

$$x = \frac{\frac{(a^a b^b 2\pi\sqrt{ab})^{\left(\frac{1}{a+b}\right)} 2^{\left(\frac{ab-1}{a+b}\right)}}{e}}{\frac{(2\pi\sqrt{ab})^{\left(\frac{1}{a+b}\right)}(a+b)2^{\left(\frac{ab-1}{a+b}\right)}}{e^2}} = \frac{\left(a^a b^b\right)^{\frac{1}{a+b}}}{a+b} e$$

When $a = b$, we get $x = \dfrac{a}{2a} * e = \dfrac{e}{2} \approx 1.359$. When $a << b$, as $a + b \approx b$, we get $x = \dfrac{b}{b} * e \approx e$. So our lower bound gives an improvement that varies between 1.35 to e depending upon the values of a and b.

2.2.2 A lower bound for $R(K_{a,b}, K_{a,b})$ using Lovász local lemma

We are interested in the question of existence of a monochromatic $K_{a,b}$ in any bicolouring of the edges of K_n. Since the same edge may be present in many distinct $K_{a,b}$'s, the colouring of any particular edge may effect the monochromaticity in many $K_{a,b}$'s. This gives the motivation for the use the Corollary 1 of Lovász local lemma (see [11]) to account for such dependencies in this context.

Lemma 1 *Lovász Local Lemma [11] Let $G(V, E)$ be a dependency graph for events $E_1, ...E_n$ in a probability space. Suppose that there exists $x_i \in [0, 1]$ for $1 \le i \le n$ such that*
$Pr\left[E_i\right] \le x_i \prod_{\{i,j\}\in E}(1 - x_j)$ *, then* $Pr\left[\bigcap_{i=1}^n \overline{E_i}\right] \ge \prod_{i=1}^n (1 - x_j)$.

A direct corollary of the lemma states the following.

Corollary 1 *[11] If every event E_i, $1 \le i \le m$ is dependent on at most d other events and $Pr\left[E_i\right] \le p$, and if $ep(d + 1) \le 1$, then $Pr\left[\bigcap_{i=1}^n \overline{E_i}\right] > 0$.*

Theorem 3 *If $e\left(2^{1-ab}\right)\left(ab\binom{n-2}{a+b-2}\binom{a+b-2}{b-1} + 1\right) \le 1$ then $R(K_{a,b}, K_{a,b}) > n$.*

Proof: We consider a random bicolouring of the complete graph K_n in which each edge is independently coloured red or blue with equal probability. Let S be the set of edges of an arbitrary $K_{a,b}$, and let E_S be the event that all edges in this $K_{a,b}$ are coloured monochromatically. For each such S, the probability of E_S is $P\left(E_S\right) = 2^{1-ab}$. We enumerate the sets of edges of all possible $K_{a,b}$'s as S_1,S_2,...,S_m, where $m = \binom{n}{a}\binom{n-a}{b}$ and each S_i is the set of all the edges of the $i^{\mathbf{th}}$ $K_{a,b}$. Clearly, each event E_{S_i} is mutually independent of all the events E_{S_j} from the set $I_j = \{E_{S_j} : |S_i \cap S_j| = 0\}$. We show that for each E_{S_i}, the number of events outside the set I_j satisfies the inequality $|\{E_{S_j} : |S_i \cap S_j| \geq 1\}| \leq ab\binom{n-2}{a+b-2}\binom{a+b-2}{b-1}$, as follows. Every S_j in this set shares at least one edge with S_i, and therefore such an S_j shares at least two vertices with S_i. We can choose the rest of the $a+b-2$ vertices of S_j from the remaining $n-2$ vertices of K_n, out of which we can choose $b-1$ for one partite set of S_j, and the remaining $a-1$ to form the second partite set of S_j, yielding a $K_{a,b}$ that shares at least one edge with S_i. We apply Corollary 1 to the set of events $E_{S_1}, E_{S_2}, ..., E_{S_m}$, with $p = 2^{1-ab}$ and $d = ab\binom{n-2}{a+b-2}\binom{a+b-2}{b-1}$, enforcing the premise $ep(d+1) \leq 1$, resulting in the lower bound for n, so that $Pr\left[\bigcap_{i=1}^{m} \overline{E}_{S_i}\right] > 0$. This non-zero probability (of none of the events E_{S_i} occurring, for $1 \leq i \leq m$) implies the existence of some bicolouring of the edges of K_n with no monochromatic $K_{a,b}$, thereby establishing the theorem. □

Solving the inequality in the statement of Theorem 3, we can compute lower bounds for $R(K_{a,b}, K_{a,b})$, for natural numbers a and b. Such lower bounds for some larger values of a and b show significant improvements over the bounds computed using Theorem 2 (see Table 1). Simplifying the inequality in the statement of Theorem 3, we get the following lower bound for $R(K_{a,b}, K_{a,b})$.

$$R(K_{a,b}, K_{a,b}) > \left(\frac{\left((a-1)^{(a-1)}(b-1)^{(b-1)} 2\pi \sqrt{(a-1)(b-1)}\right)}{eab} \right)^{\left(\frac{1}{a+b-2}\right)} \frac{2^{\left(\frac{ab-1}{a+b-2}\right)}}{e}.$$

3 The unbalanced off-diagonal case: $R(K_{a,b}, K_{c,d})$

$R(K_{a,b}, K_{c,d})$ is the minimum number n so that any n-vertex simple undirected graph G must contain a $K_{a,b}$ or its complement G' must contain the complete bipartite graph $K_{c,d}$. Equivalently, $R(K_{a,b}, K_{c,d})$ is the minimum number n such that any 2-coloring of the edges of an n-vertex complete undirected graph would contain a monochromatic $K_{a,b}$ or a monochromatic $K_{c,d}$.

3.1 A constructive lower bound for $R(K_{2,b}, K_{2,d})$

Now we present a constructive lower bound as follows by designing an explicit construction.

Theorem 4 $R(K_{2,b}, K_{2,d}) > b + d + 1$, *for all integers* $d \geq b \geq 2$.

Proof: For $d \geq b \geq 2$, we demonstrate the existence of a $K_{2,b}$-free graph with $b + d + 1$ vertices, such that its complement graph does not contain any $K_{2,d}$. The construction is illustrated for specific values of b and d in Figure 2. We have the following three exhaustive cases.

Case 1:

If $b = 2m$ for an integer m, then arrange all the vertices around a circle, numbering vertices as $v_0, v_1, v_2, ..., v_{b+d}$, and connect each vertex to its m nearest

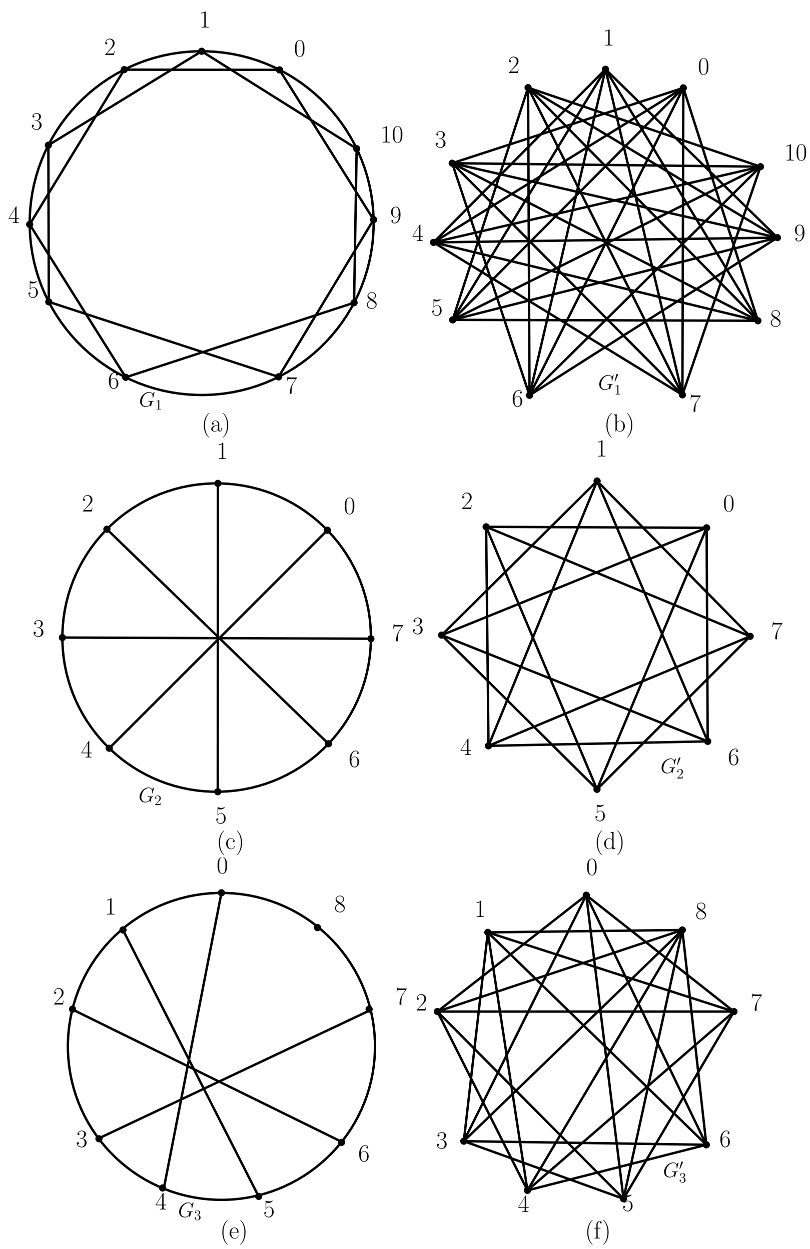

Figure 2: (i) $R(K_{2,4}, K_{2,6}) > 11$: (a) graph G_1 is $K_{2,4}$-free, and and (b) graph G'_1 is $K_{2,6}$-free, (ii) $R(K_{2,3}, K_{2,4}) > 8$: (c) graph G_2 is $K_{2,3}$-free, and and (d) graph G'_2 is $K_{2,4}$-free, (iii) $R(K_{2,3}, K_{2,5}) > 9$: (e) graph G_3 is $K_{2,3}$-free, and (f) graph G'_3 is $K_{2,5}$-free.

neighbours in clockwise (as well as counterclockwise) directions along the circle. See graph G_1 in Figure 2(a) for an example with $b = 4$ and $d = 6$. Observe that the constructed graph G is b-regular, and its complement graph is therefore d-regular. We claim that G does not have a $K_{2,b}$ since no two vertices in G share more than $b - 2$ neighbours.

We first show that for all i, $0 \leq i \leq b + d$, the vertex v_i shares exactly $2(m - 1) = b - 2$ neighbours with v_{i+1}. Here and henceforth, all arithmetic operations on indices of vertices are modulo $b + d + 1$. There are exactly $m - 1$ neighbours common to v_i and v_{i+1} in the clockwise (respectively, counterclockwise) direction of v_i (v_{i+1}), resulting in a total of $2(m - 1)$ common vertices. Similarly, the number of vertices shared by v_i with its neighbouring clockwise vertex v_{i-1} is also $b - 2$. Now consider the remaining counterclockwise neighbours v_{i+k} of v_i in G, $2 \leq k \leq m$. Observe that vertices v_i and v_{i+k} share exactly $2(m - k) + (k - 1) = 2m - k - 1 = b - k - 1$ neighbours; $m - k$ vertices clockwise (respectively, counterclockwise) of v_i (respectively, v_{i+k}), and $k - 1$ vertices clockwise of v_{i+1} and counterclockwise of v_i. So, the total number of shared neighbours between v_i and v_{i+k} (and symmetrically, between v_i and v_{i-k}), is certainly no more than $2(m - 1) = b - 2$. For the d non-adjacent vertices v_j of v_i, clearly v_j and v_i do not share more than $m < b - 2$ common neighbours. This implies that the graph G is $K_{2,b}$-free.

Now consider the complement graph G' of G. Since we have $b + d + 1$ vertices, the complement graph G' is d-regular if and only if the graph G is b-regular. See Figure 2(b) for the complement graph G'_1 of G_1, for $b = 4$ and $d = 6$. The complement graph G' can have a $K_{2,d}$ only if two vertices share all their neighbours. Each pair of vertices differ in at least two vertices in their neighbourhood in G, since any pair of two vertices can share at most $b - 2$ vertices in the b-regular graph G. This ensures that no two vertices can have all neighbours common in G'. For any vertex pair (v_i,v_j), even if the neighbourhood of v_i includes v_j, v_i still has some neighbour v_k that is not a neighbour of v_j in G, and (similarly) v_j has some neighbour v_l that is not a neighbour of v_i in G. In G' therefore, v_k is a neighbour of v_j but not a neighbour of v_i, and v_l is a neighbour of v_i but a neighbour of v_j. Therefore, G' is $K_{2,d}$-free.

Case 2:

If $b = 2m + 1$ for an integer m, and $b + d + 1$ is even (i.e., d is even), then arrange and name the vertices around a circle as in Case 1, and connect each vertex to its m nearest neighbours in counterclockwise as well as clockwise directions around the circle. Also, connect each vertex v_i to the vertex $v_{i+\frac{b+d+1}{2}}$, directly opposite to it on the circle; note that no two vertices share such a common directly opposite neighbour. The resulting graph G is b-regular. As shown in Case 1, this graph G does not have any $K_{2,b}$ as no two vertices share more than $2(m-1) = b-3 < b-2$ common neighbours. The complement graph G' is again d-regular, as in Case 1. The construction is illustrated for the case of $R(K_{2,3}, K_{2,4})$ in Figure 2(c) and (d). The only way G' can have a $K_{2,d}$ is if two vertices share all their neighbours in G'. Since two vertices G share less than $b - 2$ vertices in G, they cannot have all neighbours common in G'. This

can be shown in a manner similar to that in Case 1. So, G' is $K_{2,d}$-free.
Case 3:

If $b = 2m+1$ for some integer m, and $b+d+1$ is odd (i.e., d is odd), then arrange and name the vertices around a circle as in Cases 1 and 2, and connect (i) each vertex to its m nearest neighbours in counterclockwise as well as clockwise directions, and (ii) connect each vertex v_i to vertex $v_{i+\lfloor \frac{b+d+1}{2} \rfloor}$, for all i, $1 \le i \le \lfloor \frac{b+d+1}{2} \rfloor - 1$. This results in a graph G with $b+d$ vertices of degree b and one vertex v_{b+d} of degree $b-1$. Observe that as in Cases 1 and 2, the number of common neighbours for any two vertices in G is no more than $2(m-1) = b-3 < b-2$. This graph G is therefore free from any $K_{2,b}$.

We now show that G' is $K_{2,d}$-free. Observe that every vertex of the complement graph G' has degree d, except v_{b+d} whose degree is $d+1$. The construction is illustrated for the case of $R(K_{2,3}, K_{2,5})$ in Figure 2(e) and (f). The only way G' can have a $K_{2,d}$ is (i) if some d-degree vertex shares all its neighbours with some other d-degree vertex in G' (as in Cases 1 and 2), or (ii) if any d of the $d+1$ neighbours of the $d+1$-degree vertex v_{b+d}, are shared with a d-degree vertex in G'. Two d-degree vertices disagreeing on at least two neighbours cannot yield a $K_{2,d}$, as seen in Cases 1 and 2. So, we need to consider only the later case involving vertex v_{b+d}, whose degree is $d+1$ in G'. Consider a d-degree vertex v_i of G' and the vertex v_{b+d}. Since these two vertices share at most $b-2$ vertices in G, there is at least one neighbouring vertex v_j of v_{b+d} in G, that is not a common neighbour in G for v_i and v_{b+d}. So, v_j not connected to v_i in G and therefore a v_j is a neighbour of v_i in G'. Also, v_j is connected to v_{b+d} in G and therefore not a neighbour of v_{b+d} in G'. So, G' does not have a $K_{2,d}$ where v_i and v_{b+d} should share d neighbours. □

Now we derive a lower bound on such numbers using the probabilistic method.

3.2 A probabilistic lower bound for $R(K_{a,b}, K_{c,d})$

Theorem 5 *For all $n \in N$ and $0 < p < 1$, if*

$$\binom{n}{a}\binom{n-a}{b}p^{ab} + \binom{n}{c}\binom{n-c}{d}(1-p)^{cd} < 1 \ , \tag{2}$$

then $R(K_{a,b}, K_{c,d}) > n$.

Proof: Consider a random bicolouring of the edges of K_n with colours red and blue with probabilities p and $1-p$, respectively. The probability that a particular red $K_{a,b}$ exists is p^{ab}. So, the probability that any red $K_{a,b}$ exists is $\binom{n}{a}\binom{n-a}{b}p^{ab}$. Similarly, the probability that a particular blue $K_{c,d}$ exists is $(1-p)^{cd}$, and the probability that any red $K_{c,d}$ exists is $\binom{n}{c}\binom{n-c}{d}(1-p)^{cd}$. So, the probability that the bicoloured K_n contains any red $K_{a,b}$ or any blue $K_{c,d}$ is $\binom{n}{a}\binom{n-a}{b}p^{ab} + \binom{n}{c}\binom{n-c}{d}(1-p)^{cd}$. The theorem follows by setting this probability to less than unity. □

Corollary 2 *For all $n \in N$ and $0 < p < 1$, if $\binom{n}{a}\binom{n-a}{a}p^{a^2} + \binom{n}{b}\binom{n-b}{b}(1-p)^{b^2} < 1$, then $R(K_{a,a}, K_{b,b}) > n$.*

3.3 A lower bound for $R(K_{a,b}, K_{c,d})$ using Lovász local lemma

We are interested in the question of existence of a monochromatic $K_{a,b}$ or a monochromatic $K_{c,d}$ in any bicolouring of the edges of K_n. Since the same edge may be shared by many distinct $K_{a,b}$'s and $K_{c,d}$'s, the colouring of any particular edge may affect the monochromaticity in many $K_{a,b}$'s and $K_{c,d}$'s. This gives the motivation for the use of the Corollary 1 of Lovász local lemma in this context.

Table 2: Lower bounds for $R(K_{a,b}, K_{c,d})$ from Inequality 2 (left) and Theorem 6 (right)

c,d a,b	10,11	10,12	10,13	11,12	11,13	12,13	12,14	13,14	14,15	15,16
10,11	182,179 0.5,0.5	200,194 0.49,0.49	215,207 0.48,0.48	222,217 0.48,0.48	241,232 0.47,0.47	266,256 0.46,0.47	285,289 0.46,0.46	317,324 0.45,0.45	376,373 0.43,0.44	446,436 0.42,0.42
10,12	200,194 0.51,0.51	220,218 0.5,0.5	238,233 0.49,0.49	245,245 0.49,0.49	268,262 0.48,0.48	296,288 0.47,0.47	316,316 0.46,0.47	350,359 0.46,0.46	423,410 0.44,0.45	498,504 0.43,0.43
10,13	215,207 0.52,0.52	238,233 0.51,0.51	261,263 0.5,0.5	266,263 0.5,0.5	294,297 0.49,0.49	327,327 0.48,0.48	353,345 0.47,0.47	384,385 0.46,0.47	471,471 0.45,0.45	542,577 0.44,0.44
11,12	222,217 0.52,0.52	245,245 0.51,0.51	266,263 0.5,0.5	275,277 0.5,0.5	300,297 0.49,0.49	333,327 0.48,0.48	355,358 0.48,0.48	398,409 0.47,0.47	482,471 0.45,0.45	573,584 0.5,0.44
11,13	241,232 0.53,0.53	268,262 0.52,0.52	294,297 0.51,0.51	300,297 0.51,0.51	332,337 0.5,0.5	370,373 0.49,0.49	399,394 0.48,0.48	435,440 0.47,0.48	540,544 0.46,0.46	628,669 0..45,0.45
11,14	256,256 0.53,0.53	286,279 0.53,0.52	318,312 0.52,0.52	320,315 0.52,0.51	358,355 0.51,0.51	402,405 0.50,0.50	443,451 0.49,0.49	489,490 0.48,0.48	586,613 0.47,0.47	690,709 0.45,0.46
12,13	266,256 0.54,0.53	296,288 0.53,0.53	327,327 0.52,0.52	333,327 0.52,0.52	370,373 0.51,0.51	415,426 0.5,0.5	450,451 0.49,0.49	492,490 0.48,0.48	611,629 0.47,0.47	708,752 0.46,0.46
12,14	285,289 0.54,0.54	316,316 0.54,0.53	353,345 0.53,0.53	355,358 0.52,0.52	399,394 0.52,0.52	450,451 0.51,0.51	500,518 0.5,0.5	555,565 0.49,0.49	662,690 0.5,0.48	793,801 0.46,0.47
12,15	315,312 0.55,0.55	348,356 0.54,0.54	376,385 0.53,0.53	395,408 0.53,0.53	431,440 0.51,0.52	481,490 0.51,0.51	535,540 0.51,0.51	606,622 0.50,0.50	722,732 0.48,0.48	895,927 0.47,0.47
13,14	317,324 0.55,0.55	350,359 0.54,0.54	384,385 0.54,0.53	398,409 0.53,0.53	435,440 0.53,0.52	492,490 0.52,0.51	555,565 0.51,0.51	623,652 0.5,0.5	730,757 0.49,0.49	910,928 0.47,0.47
13,15	348,337 0.56,0.56	387,386 0.55,0.55	423,439 0.54,0.54	441,443 0.54,0.54	485,504 0.53,0.53	544,563 0.52,0.52	588,598 0.52,0.51	670,683 0.51,0.51	827,852 0.49,0.49	1010,1078 0.48,0.48
14,15	376,373 0.57,0.56	423,410 0.56,0.55	471,471 0.55,0.55	482,471 0.55,0.55	540,544 0.54,0.54	611,629 0.53,0.53	662,690 0.52,0.52	730,757 0.51,0.51	935,994 0.5,0.5	1105,1166 0.49,0.49
14,16	401,417 0.57,0.57	444,468 0.56,0.56	500,503 0.56,0.55	509,537 0.55,0.55	571,580 0.55,0.54	652,650 0.54,0.53	737,753 0.53,0.53	828,878 0.52,0.52	995,1029 0.51,0.51	1226,1280 0.49,0.49
15,16	446,436 0.58,0.58	498,504 0.57,0.57	542,577 0.56,0.56	573,584 0.56,0.56	628,669 0.55,0.55	708,752 0.54,0.54	793,801 0.54,0.53	910,928 0.53,0.53	1105,1166 0.51,0.51	1399,1509 0.51,0.50

Theorem 6 *If for some* $0 < p < 1$, $\left\{ab\binom{n-2}{a-1}\binom{n-a-1}{b-1} + 1\right\} p^{ab} e^{1+\frac{ab}{cd}} \leq 1$ *and* $\left\{cd\binom{n-2}{c-1}\binom{n-c-1}{d-1} + 1\right\} e^{-pcd} e^{1+\frac{cd}{ab}} \leq 1$, *then* $R(K_{a,b}, K_{c,d}) > n$.

Proof: We consider a random bicolouring of the complete graph K_n in which each edge is independently coloured red or blue with probabilities p and $(1-p)$ respectively. Let S be the set of edges of an arbitrary $K_{a,b}$,T be the set of edges of an arbitrary $K_{c,d}$, . let E_S be the event that all edges in the $K_{a,b}$ S are coloured monochromatically red and let E_T be the event that all edges in the $K_{c,d}$ T are coloured monochromatically blue. For each such S, the probability of E_S is $P(E_S) = p^{ab}$. Similarly For each such T, the probability of E_T is $P(E_T) = (1-p)^{cd}$. We enumerate the sets of edges of all possible $K_{a,b}$'s and

$K_{c,d}$'s as A_1,A_2,...,A_m, where $m = \binom{n}{a}\binom{n-a}{b} + \binom{n}{c}\binom{n-c}{d}$. Clearly, each event E_{A_i} is mutually independent of all the events E_{A_j} from the set $\{E_{A_j} : |A_i \cap A_j| = 0\}$; since for any such A_j , A_i and A_j share no edges. Now again as the events can be a monochromatic $K_{a,b}$ or $K_{c,d}$, Let A_{ab} denote a $K_{a,b}$ and A_{cd} denote a $K_{c,d}$.

For each $E_{A_{ab}}$, the number of events outside this set satisfies the inequality $|\{E_{A_j} : |A_{ab} \cap A_j| \geq 1\}| \leq ab\{\binom{n-2}{a-1}\binom{n-a-1}{b-1} + \binom{n-2}{c-1}\binom{n-c-1}{d-1}\}$; every A_j in this set shares at least one edge with A_{ab}, and therefore such an A_j shares at least two vertices with A_{ab}. If this A_j is a $K_{a,b}$, then We can choose the rest of the $a+b-2$ vertices of A_j from the remaining $n-2$ vertices of K_n, out of which we can choose $a-1$ for one partite set of A_j, and the remaining $b-1$ to form the second partite set of A_j, yielding a $K_{a,b}$ that shares at least one edge with A_{ab}. On the other hand, if this A_j is a $K_{c,d}$, then We can choose the rest of the $c+d-2$ vertices of A_j from the remaining $n-2$ vertices of K_n, out of which we can choose $c-1$ for one partite set of A_j, and the remaining $d-1$ to form the second partite set of A_j, yielding a $K_{a,b}$ that shares at least one edge with A_{ab}. Similarly, For each $E_{A_{cd}}$, the number of events that shares atleast one edge satisfies the inequality $|\{E_{A_j} : |A_{cd} \cap A_j| \geq 1\}| \leq cd\{\binom{n-2}{a-1}\binom{n-a-1}{b-1} + \binom{n-2}{c-1}\binom{n-c-1}{d-1}\}$. By applying Theorem 1, we want to show that

$$Pr\left[\bigcap_{i=1}^{m} \overline{E}_{A_i}\right] > 0. \tag{3}$$

This non-zero probability (of none of the events E_{A_i} occurring, for $1 \leq i \leq m$) implies the existence of some bicolouring of the edges of K_n with no red $K_{a,b}$ or blue $K_{c,d}$, thereby establishing the theorem. The Inequality 3 is satisfied if the following conditions hold.

$$\begin{aligned} Pr\left[E_{A_{ab}}\right] &\leq x_{ab}\left(1-x_{ab}\right)^{ab\binom{n-2}{a-1}\binom{n-a-1}{b-1}}\left(1-x_{cd}\right)^{ab\binom{n-2}{c-1}\binom{n-c-1}{d-1}} \\ Pr\left[E_{A_{cd}}\right] &\leq x_{cd}\left(1-x_{ab}\right)^{cd\binom{n-2}{a-1}\binom{n-a-1}{b-1}}\left(1-x_{cd}\right)^{cd\binom{n-2}{c-1}\binom{n-c-1}{d-1}}, \\ &\qquad\qquad \text{for some } x_{ab}, x_{cd}. \end{aligned} \tag{4}$$

Choosing $x_{ab} = \frac{1}{ab\binom{n-2}{a-1}\binom{n-a-1}{b-1}+1}$, $x_{cd} = \frac{1}{cd\binom{n-2}{c-1}\binom{n-c-1}{d-1}+1}$ and using the inequalities $(1-p)^{cd} \leq e^{-pcd}$ and $\left(1 - \frac{1}{d+1}\right)^{d} \geq e$, we get

$$\begin{aligned} &\left\{ab\binom{n-2}{a-1}\binom{n-a-1}{b-1}+1\right\}p^{ab}e^{1+\frac{ab}{cd}} \leq 1, \text{ and} \\ &\left\{cd\binom{n-2}{c-1}\binom{n-c-1}{d-1}+1\right\}e^{-pcd}e^{1+\frac{cd}{ab}} \leq 1. \end{aligned} \tag{5}$$

To get a lower bound on $R(K_{a,b}, K_{c,d})$, we choose the largest value of n, such that both of these conditions are satisfied. □

Solving the inequality in the statement of Theorem 6, we can compute lower bounds for $R(K_{a,b}, K_{c,d})$, for natural numbers a, b, c and d. Such lower bounds for some larger values of the arguments a, b, c and d show significant improvements over the bounds computed using Theorem 5 (see Table 2). These lower bounds in Table 2 are computed using the inequalities in Theorems 5 and 6; this is done by incrementing the value of the probability parameter p by the hundredths of a decimal and determining the largest resulting lower bounds from the inequalities for each set of values for the arguments a, b, c and d. The values of such probabilities are tabulated below the corresponding lower bound entries in the table.

Corollary 3 *If for some* $0 < p < 1$, $\left\{a^2\binom{n-2}{a-1}\binom{n-a-1}{a-1}+1\right\}p^{a^2}e^{1+\frac{a^2}{b^2}} \leq 1$ *and* $\left\{b^2\binom{n-2}{b-1}\binom{n-b-1}{b-1}+1\right\}e^{-pb^2}e^{1+\frac{b^2}{a^2}} \leq 1$, *then* $R(K_{a,a}, K_{b,b}) > n$.

4 Lower bounds for Ramsey numbers for complete tripartite 3-uniform subgraphs

Let $R'(a, b, c)$ be the minimum number n such that any n-vertex 3-uniform hypergraph $G(V, E)$, or its complement $G'(V, E)$ contains a $K_{a,b,c}$. An r-uniform hypergraph is a hypergraph where every hyperedge has exactly r vertices. (Hyperedges of a hypergraph are subsets of the vertex set. So, usual graphs are 2-uniform hypergraphs.) Here, $K_{a,b,c}$ is defined as the complete tripartite 3-uniform hypergraph with vertex set $A \cup B \cup C$, where the A, B and C have a, b and c vertices respectively, and $K_{a,b,c}$ has abc 3-uniform hyperedges $\{u, v, w\}$, $u \in A$, $v \in B$ and $w \in C$. It is easy to see that $R'(1, 1, 1) = 3$; with 3 vertices, there is one possible 3-uniform hyperedge which either is present or absent in G.

Theorem 7 $R'(1, 1, 2) = 4$.

Proof: Consider the complete 3-uniform hypergraph with vertex set $V = \{1, 2, 3, 4\}$ and set of exactly four hyperedges $H = \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}\}$. Since vertex 1 is present in 3 hyperedges, any (empty or non-empty) subset S of H, or its complement $H \setminus S$ must contain at least two hyperedges containing the vertex 1. Observe that any such set of two hyperedges is a $K_{1,1,2}$. □

The fact that $R'(1, 1, 3) > 5$ can be established by the counterexample given in Figure 3, where neither the 3-uniform hypergraph G nor its complement G' has a $K_{1,1,3}$. The vertices are v_1, v_2, v_3, v_4, v_5 and $e_1, e_2..., e_{10}$ represent all the ten possible 3-uniform hyperedges. The hypergraph G has five hyperedges viz., $e_1\,(\{1, 2, 3\})$, $e_2\,(\{1, 2, 4\})$, $e_3\,(\{1, 3, 5\})$, $e_4\,(\{2, 4, 5\})$, $e_5\,(\{3, 4, 5\})$. The complement hypergraph G' has the remaining five hyperedges, viz., $e_6\,(\{1, 2, 5\})$, $e_7\,(\{1, 3, 4\})$, $e_8\,(\{1, 4, 5\})$, $e_9\,(\{2, 3, 4\})$, $e_{10}\,(\{2, 3, 5\})$.

We also show that $R'(1, 1, 4) > 6$. We found the following counterexample. Consider the set $E = \{\{1, 2, 4\}, \{1, 3, 5\}, \{1, 3, 6\}, \{1, 4, 5\}, \{1, 4, 6\}, \{1, 5, 6\}, \{2,$

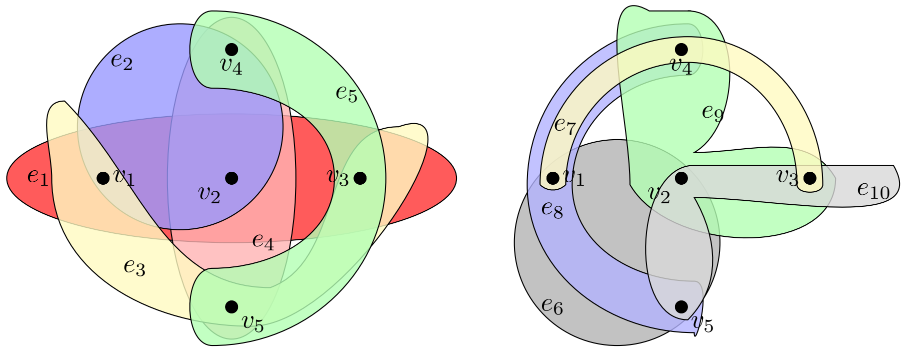

Figure 3: Hypergraph G (left) and its complement G' (right). Neither G nor G' has a $K_{1,1,3}$

$3,4\}, \{2,3,5\}, \{2,3,6\}, \{2,4,5\}\}$ of hyperedges of a 6-vertex 3-uniform hypergraph G. The set $E' = \{\{1,2,3\}, \{1,2,5\}, \{1,2,6\}, \{1,3,4\}, \{2,4,6\}, \{2,5,6\}, \{3,4,5\}, \{3,4,6\}, \{3,5,6\}, \{4,5,6\}\}$ is the set of of hyperedges of the complement hypergraph G' of G. Note that neither G nor G' has a $K_{1,1,4}$.

The example in Figure 3 showing $R'(1,1,3) > 5$ was discovered using the following method; we have used the same method also for showing that $R(1,1,4) \geq 7$. As there are $\binom{5}{3} = 10$ distinct 3-uniform hyperedges possible with 5 vertices. So, there are 2^{10} possible 3-uniform hypergraphs. We designated each of the 10 hyperedges with a distinct number starting from 0 to 9. For example, hyperedge $\{1,2,3\}$ is mapped to 0 and $\{3,4,5\}$ is mapped to 9. Then, we generated every distinct $K_{1,1,3}$, which are $\binom{5}{3} = 10$ in number. We generated all the possible 2^{10} hypergraphs and checked for the existence of each $K_{1,1,3}$. For example, the hyperedges $\{\{1,2,3\}, \{1,2,4\}, \{1,2,5\}\}$, numbered as 0, 1 and 2, respectively, constitute a $K_{1,1,3}$ denoted as (0 1 2), and the hyperedges $\{\{1,2,3\}, \{1,3,4\}, \{1,3,5\}\}$ constitute a $K_{1,1,3}$ denoted as (0 4 5). For generating all possible hypergraphs, we take a 10-bit binary number, where each bit represents a particular hyperedge (the 0^{th} bit represents $\{1,2,3\}$, and the 9^{th} bit represents $\{3,4,5\}$), and generate all its possible combinations. Now for every 10-bit binary string, we check for the existence of any $K_{1,1,3}$. For example, let the binary string be 000000111. This string represents the hypergraph with edges $\{\{1,2,3\}, \{1,2,4\}, \{1,2,5\}\}$ denoting the presence of $K_{1,1,3}$ denoted by (0 1 2). If for any hypergraph, no $K_{1,1,3}$ is present, then we check the existence of a $K_{1,1,3}$ in the complement hypergraph. If neither the hypergraph nor its complement have a $K_{1,1,3}$, then we get our sought counterexample hypergraph. Determining such Ramsey numbers for higher parameters by exhaustive searching using computer programs is computationally very expensive in terms of running time.

We have the following upper bound for $R'(1,1,b)$.

Theorem 8 $R'(1,1,b) \leq 2b+1$.

Proof: Let v_1,v_2, ..., v_{2b+1} be the $2b+1$ vertices. Then, for any pair of vertices v_i, v_j, there are $2b-1$ possible 3-uniform hyperedges (each hyperedge containing one distinct vertex from the remaining $2b-1$ vertices). So, by the pigeonhole principle, either the graph or its complement must include b of these hyperedges containing both v_i and v_j. This set of b hyperedges denotes a $K_{1,1,b}$. □

Based on our findings $R(1,1,3) \geq 6$ (see Figure 3), and $R(1,1,4) \geq 7$, we state our conjecture for $R'(1,1,b), b \geq 3$, as follows,

Conjecture 1 *$R'(1,1,b) \geq 2b$.*

Note that settling this conjecture positively would require showing that for some $(2b-1)$-vertex 3-uniform hypergraph G, neither G nor G' has a $K_{1,1,b}$. We related this problem to that of the existence of a t-design. A t-design is defined as follows. A t-(v,k,λ) design is an *incidence structure* of *points* and *blocks* with properties (i) v is the number of points, (ii) each block is incident on k points, and (iii) each subset of t points is incident on λ common blocks [1].

Lemma 2 *If there is a $2-(2b-1,3,b-1)$ design then $R'(1,1,b) \geq 2b$.*

Proof: The existence of 2-($2b-1$,3,$b-1$) design would suggest that there exist a 3-uniform hypergraph with $2b-1$ vertices such that every pair of vertices forms a hyperedge with exactly $b-1$ other vertices. This implies that the hypergraph is free of $K_{1,1,b}$. So, every pair of vertices will also form a hyperedge in the complement hypergraph with exactly $(2b-1)-2-(b-1) = b-2$ vertices. Therefore, the complement hypergraph is also free of $K_{1,1,b}$. □

Table 3: Lower bounds for $R'(a,a,a)$ by Theorem 9 (left) and Theorem 10 (right)

a	3	4	5	6	7	8
$R'(a,a,a)$	14,19	84,138	800,1765	11773,35167	269569,1073543	9650620,50616072

Table 4: Lower bounds for $R'(a,b,c)$ by Theorem 9 (left) and Theorem 10 (right)

	a=2	a=3	a=3	a=3	a=4	a=4	a=5	a=6	a=6	a=6	a=6
c	5	3	4	5	4	5	5	2	3	4	5
b											
2	9,13	8,11	11,16	16,22	18,25	26,36	40,58	11,16	21,29	36,52	59,87
3	16,22	14,19	23,32	35,50	41,61	68,107	124,208		50,74	107,175	209,371
4	26,36		41,61	68,107	84,138	159,281	334,653			277,521	643,1354
5	40,58			124,208		334,653	800,1765				1740,4194

4.1 Probabilistic lower bound for $R'(a,b,c)$

Theorem 9 $R'(a,b,c) > \frac{\left(a^a b^b c^c \sqrt{(2\pi)^3 abc}\right)^{\left(\frac{1}{a+b+c}\right)} 2^{\left(\frac{abc-1}{a+b+c}\right)}}{e}.$

Proof: Consider the probability of existence of a particular $K_{a,b,c}$ in G or G', where G is a 3-uniform hypergraph and G' is its complement. The sum p of such probabilities over all possible distinct $K_{a,b,c}$'s is an upper bound on the probability that some $K_{a,b,c}$ exists in G or G'. Let n be the number of vertices of hypergraph G. As in the proof of Theorem 2, we observe that the number of $K_{a,b,c}$'s is no more than $\binom{n}{a}\binom{n-a}{b}\binom{n-a-b}{c}$. Each $K_{a,b,c}$ has exactly abc hyperedges. Each hyperedge can be present in G or G' with equal probability. So, the probability that all hyperedges of a particular $K_{a,b,c}$ are in G is $\left(\frac{1}{2}\right)^{abc}$. Therefore, the probability that a particular $K_{a,b,c}$ is present in either G or G' is $2\left(\frac{1}{2}\right)^{abc} = 2^{1-abc}$. So, the probability p that some $K_{a,b,c}$ is either in G or in G', is $\binom{n}{a}\binom{n-a}{b}\binom{n-a-b}{c}2^{1-abc}$. Using $n > \frac{\left(a^a b^b c^c \sqrt{(2\pi)^3 abc}\right)^{\left(\frac{1}{a+b+c}\right)} 2^{\left(\frac{abc-1}{a+b+c}\right)}}{e}$ and Stirling's approximation as in the proof of Theorem 2, we get $p < 1$, thereby ensuring the existence of a hypergraph G of n vertices such that neither G nor G' has a $K_{a,b,c}$. For details, see [10]. □

See Tables 3 and 4 for some computed lower bounds based on Theorem 9.

4.2 A lower bound for $R'(a,b,c)$ using Lovász local lemma

Theorem 10 *If* $e\left(2^{1-abc}\right)\left(abc\binom{n-3}{a+b+c-3}\binom{a+b+c-3}{b-1}\binom{a+c-2}{c-1}+1\right) \leq 1$ *then* $R'(a,b,c) > n$.

Proof: We perform analysis as done earlier in Section 2.2.2. Consider a random bicoloring of the hyperedges of the complete 3-uniform hypergraph of n vertices, in which each hyperedge is independently colored red or blue with equal probability. Let S be the set of hyperedges of an arbitrary $K_{a,b,c}$, and let E_S be the event that the $K_{a,b,c}$ is coloured monochromatically. For each such S, $P\left(E_S\right) = 2^{1-abc}$. If we enumerate all possible $K_{a,b,c}$'s as S_1,S_2,...,S_m, where $m = \binom{n}{a}\binom{n-a}{b}\binom{n-a-b}{c}$, and each S_i is the set of all the hyperedges of the $i^{\mathbf{th}}$ $K_{a,b,c}$, then each event E_{S_i} is mutually independent of all the events from the set $I_j = \{E_{S_j} : |S_i \cap S_j| = 0\}$. We claim that for each E_{S_i}, the number of events outside the set I_j satisfies the inequality $\{E_{S_j} : |S_i \cap S_j| \geq 1\} \leq abc\binom{n-3}{a+b+c-3}\binom{a+b+c-3}{b-1}\binom{a+c-2}{c-1}$, as follows. Every S_j in this set shares at least one of the abc hyperedges of S_i, and therefore S_j shares at least three vertices with S_i. We can choose the rest of the $a+b+c-3$ vertices of S_j from the remaining $n-3$ vertices, out of which we can choose $b-1$ for the second partite set of S_j, and the remaining $c-1$ for the third partite set of S_j, thereby yielding a $K_{a,b,c}$ which shares at least one hyperedge edge with S_i. Applying Corollary 1 to the set of events E_{S_1},E_{S_2},...,E_{S_m}, with $p = 2^{1-abc}$ and $d = abc\binom{n-3}{a+b+c-3}\binom{a+b+c-3}{b-1}\binom{a+c-2}{c-1}$ yields $ep(d+1) \leq 1$, implying $Pr\left[\bigcap_{i=1}^{m} \overline{E}_{S_i}\right] > 0$. Since no event E_{S_i} occurs for some random bicoloring of the hyperedges, no monochromatic $K_{a,b,c}$ exists in that bicoloring. This establishes the theorem. □

See Tables 3 and 4 for some computed lower bounds based on Theorem 10; the values based on Theorem 10 to the right in each cell of these tables are much better than those based on Theorem 9, to the left in the respective cells.

5 Concluding remarks

The probabilistic method is useful in establishing lower bounds for Ramsey numbers. It is worthwhile studying the application of Lovász local lemma, possibly more effectively and accurately, so that higher lower bounds may be determined. In our work we have considered bicolorings of K_n and the existence of monochromatic complete bipartite subgraphs ($K_{a,b}$ in the unbalanced diagonal case, $K_{a,b}$ or $K_{c,d}$ in the unbalanced off-diagonal case) in arbitrary bicolorings of the edges of K_n; some authors consider bicolorings of $K_{n,n}$ instead of bicolorings of K_n, and derive bounds for corresponding Ramsey numbers. For values and bounds on such Ramsey numbers see [12]. For computing the lower bounds in Tables 1, 2, 3 and 4, we have used computer programs. The code for these programs are available from the authors on request. As the sizes of the complete bipartite graphs (tripartite 3-uniform hypergraphs) grow, the computation time required for computing the lower bounds becomes prohibitive.

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