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2-Layer Graph Drawings with Bounded Pathwidth

David R. Wood[1]

[1]School of Mathematics, Monash University
Melbourne, Australia

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Abstract. This paper determines which properties of 2-layer drawings characterise bipartite graphs of bounded pathwidth.

1 Introduction

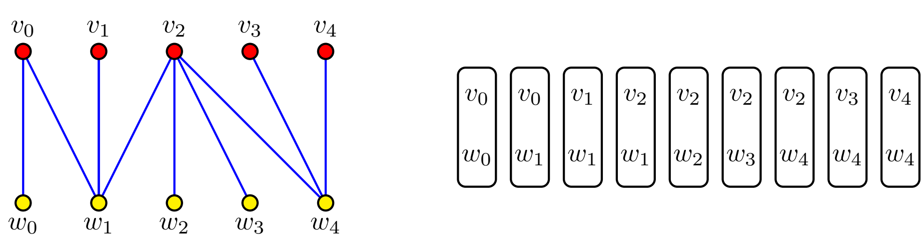

Figure 1: A caterpillar drawn on 2-layers with no crossings, and the corresponding path-decompostion with width 1.

A *2-layer drawing* of a bipartite graph G with bipartition $\{A, B\}$ positions the vertices in A at distinct points on a horizontal line, and positions the vertices in B at distinct points on a different horizontal line, and draws each edge as a straight line-segment. 2-layer graph drawings are of fundamental importance in graph drawing research and have been widely studied [2, 6, 7, 10, 11, 14–17, 19, 21, 22, 24]. As illustrated in Figure 1, the following basic connection between 2-layer graph drawings and graph pathwidth[1] is folklore:

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E-mail address: david.wood@monash.edu (David R. Wood)

[1]A *path-decomposition* of a graph G is a sequence $(B_1, \dots, B_n)$ of subsets of $V(G)$ (called *bags*), such that $B_1 \cup \dots \cup B_n = V(G)$, and for $1 \leqslant i < j < k \leqslant n$ we have $B_i \cap B_k \subseteq B_j$; that is, for each vertex v the bags containing

Observation 1 *A connected bipartite graph G has a 2-layer drawing with no crossings if and only if G is a caterpillar if and only if G has pathwidth 1.*

Motivated by this connection, we consider (and answer) the following question: what properties of 2-layer drawings characterise bipartite graphs of bounded pathwidth?

A *matching* in a graph G is a set of edges in G, no two of which are incident to a common vertex. A *k-matching* is a matching of size k. In a 2-layer drawing of a graph G, a *k-crossing* is a set of k pairwise crossing edges (which necessarily is a k-matching). Excluding a k-crossing is not enough to guarantee bounded pathwidth. For example, as illustrated in Figure 2, if T_h is the complete binary tree of height h, then T_h has a 2-layer drawing with no 3-crossing, but it is well known that T_h has pathwidth $\lfloor h/2 \rfloor + 1$. Even stronger, if G_h is the $h \times h$ square grid graph, then G_h has a 2-layer drawing with no 3-crossing, but G_h has treewidth and pathwidth h.

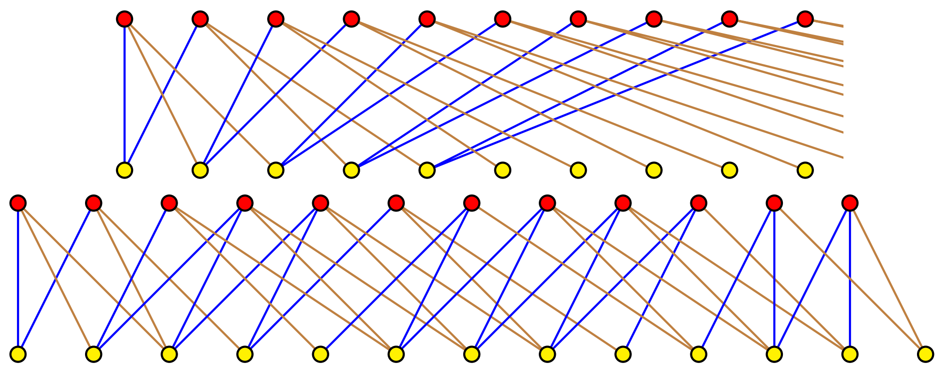

Figure 2: 2-layer drawings of a complete binary tree and a 5×5 grid. There is no 3-crossing since each edge is assigned one of two colours, so that monochromatic edges do not cross.

Angelini, Da Lozzo, Förster, and Schneck [1] showed that every graph that has a 2-layer drawing with at most k crossings on each edge has pathwidth at most $k + 1$. However, this property does not characterise bipartite graphs with bounded pathwidth. For example, as illustrated in Figure 3, if S_n is the 1-subdivision of the n-leaf star, then S_n is bipartite with pathwidth 2, but in every 2-layer drawing of S_n, some edge has at least $(n-1)/2$ crossings.

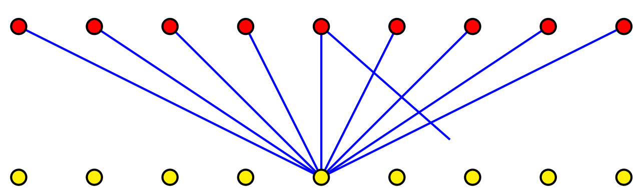

Figure 3: Every 2-layer drawing of S_9 has at least 4 crossings on some edge.

v form a non-empty sub-sequence of $(B_1, \dots, B_n)$. The *width* of a path-decomposition $(B_1, \dots, B_n)$ is $\max_i |B_i| - 1$. The *pathwidth* of a graph G is the minimum width of a path-decomposition of G. Pathwidth is a fundamental parameter in graph structure theory [4, 5, 8, 23] with many connections to graph drawing [2, 3, 10, 12, 13, 18, 20, 24]. A *caterpillar* is a tree such that deleting the leaves gives a path. It is a straightforward exercise to show that a connected graph has pathwidth 1 if and only if it is a caterpillar.

These examples motivate the following definition. A set S of edges in a 2-layer drawing is *non-crossing* if no two edges in S cross. In a 2-layer drawing of a graph G, an *(s,t)-crossing* is a pair (S,T) where S is a non-crossing s-matching, T is a non-crossing t-matching, and every edge in S crosses every edge in T; as illustrated in Figure 4.

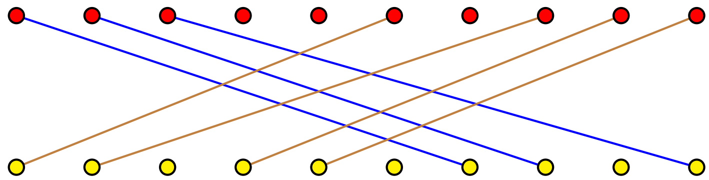

Figure 4: Example of a $(3,4)$-crossing.

We show that excluding a k-crossing and an (s,t)-crossing guarantees bounded pathwidth.

Theorem 2 *For all $k,s,t \in \mathbb{N}$, every bipartite graph G that has a 2-layer drawing with no $(k+1)$-crossing and no (s,t)-crossing has pathwidth at most $8k^2(t-1)+4k^2(s-1)^2(s-2)+5k+4$.*

We prove the following converse to Theorem 2.

Theorem 3 *For any $k \in \mathbb{N}$ every bipartite graph G with pathwidth at most k has a 2-layer drawing with no $(k+2)$-crossing and no $(k+1,k+1)$-crossing.*

Theorems 2 and 3 together establish the following rough characterisation of bipartite graphs with bounded pathwidth, thus answering the opening question.

Corollary 4 *A class $\mathcal{G}$ of bipartite graphs has bounded pathwidth if and only if there exists $k,s,t \in \mathbb{N}$ such that every graph in $\mathcal{G}$ has a 2-layer drawing with no k-crossing and no (s,t)-crossing.*

2 Proofs

We use the following notation throughout. Consider a 2-layer drawing of a bipartite graph with bipartition $\{A,B\}$. Let $\preceq_A$ be the total order of A, where $v \prec_A w$ if v is to the left of w in the drawing. Define $\preceq_B$ similarly. Let $\preceq$ be the poset on $E(G)$, where $vw \preceq xy$ if $v \preceq_A x$ and $w \preceq_B y$. Two edges of G are comparable under $\preceq$ if and only if they do not cross. Thus every chain under $\preceq$ is a set of pairwise non-crossing edges, and every antichain under $\preceq$ is a matching of pairwise crossing edges.

Lemma 5 *Let G be a bipartite graph with bipartition A,B, where each vertex in A has degree at least 1 and each vertex in B has degree at most d. Assume that G has a 2-layer drawing with no $(k+1)$-crossing and no non-crossing $(\ell+1)$-matching. Then $|A| \leqslant k\ell d$.*

Proof: Let X be a set of edges in G with exactly one edge in X incident to each vertex in A. So $|X| = |A|$. Let $E_1,\dots,E_d$ be the partition of X, where for each edge $vw \in E_i$, if $v \in A$ and $w \in B$, then v is the i-th neighbour of w with respect to $\preceq_A$. So each E_i is a matching. Since G has no $(k+1)$-crossing, every antichain in $\preceq$ has size at most k. By Dilworth's Theorem [9] applied to $\preceq$ (restricted to E_i), there is a partition $E_{i,1},\dots,E_{i,k}$ of E_i such that edges in each $E_{i,j}$ are pairwise non-crossing. By assumption, $|E_{i,j}| \leqslant \ell$. Thus $|A| = |X| \leqslant k\ell d$. □

Proof of Theorem 2: Consider a bipartite graph G with bipartition $\{A, B\}$ and a 2-layer drawing of G with no $(k+1)$-crossing and no (s,t)-crossing. Our goal is to show that $\text{pw}(G) \leqslant 8k^2(t-1) + 4k^2(s-1)^2(s-2) + 5k + 4$. (We make no effort to optimise this bound.)

Consider the partial order $\preceq$ defined above. By assumption, every antichain in $\preceq$ has size at most k. By Dilworth's Theorem [9], there is a partition of $E(G)$ into k chains under $\preceq$. Each chain is a caterpillar forest, which can be oriented with outdegree at most 1 at each vertex. So each vertex has out-degree at most k in G. For each vertex v, let $N_G^+[v] := \{w \in V(G) : \overrightarrow{vw} \in E(G)\} \cup \{v\}$, which has size at most $k+1$.

As illustrated in Figure 5, let $X = \{e_1, \dots, e_n\}$ be a maximal non-crossing matching, where $e_1 \prec e_2 \prec \cdots \prec e_n$. (Here n is not related to $|V(G)|$.) Let Y_0 be the set of vertices of G strictly to the left of e_1. For $i \in \{1, 2, \dots, n-1\}$, let Y_i be the set of vertices of G strictly between e_i and e_{i+1}. Let Y_n be the set of vertices of G strictly to the right of e_n. By the maximality of X, each set Y_i is independent. For $i \in \{0, 1, \dots, n\}$, arbitrarily enumerate $Y_i = \{v_{i,1}, \dots, v_{i,m_i}\}$. Note that $v_{i,j}$ is an end-vertex of no edge in X (for all i, j).

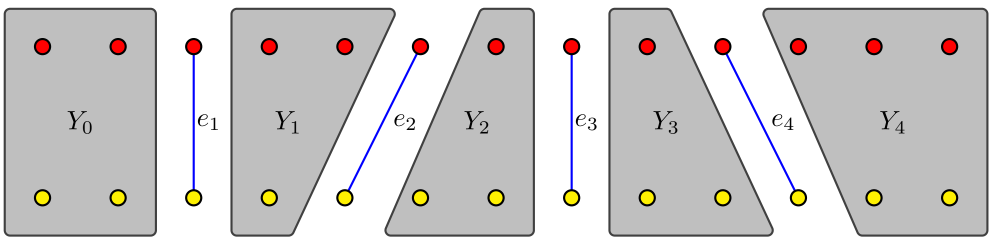

Figure 5: A maximal non-crossing matching $\{e_1, \dots, e_n\}$ and associated independent sets $Y_0, \dots, Y_n$.

As illustrated in Figure 6, for each $i \in \{1, \dots, n\}$, if $e_i = xy$ then let $N_i = N_G^+[x] \cup N_G^+[y]$. Note that $|N_i| \leqslant |N_G^+[x]| + |N_G^+[y]| \leqslant 2(k+1)$. For each $i \in \{1, \dots, n\}$, let V_i be the set consisting of N_i along with every vertex $v \in V(G)$ such that some arc $\overrightarrow{zv} \in E(G)$ crosses e_i. For each $i \in \{0, 1, \dots, n\}$ and $j \in \{1, \dots, m_i\}$, let $V_{i,j} := (V_i \cup V_{i+1}) \cup N_G^+[v_{i,j}]$ where $V_0 := V_{n+1} := \varnothing$.

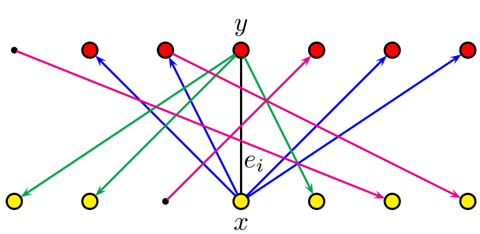

Figure 6: The set of vertices V_i where $e_i = xy$ are shown in red and yellow.

We now prove that

$$(V_{0,1}, \dots, V_{0,m_0}; V_1; V_{1,1}, \dots, V_{1,m_1}; \dots; V_n; V_{n,1}, \dots, V_{n,m_n}) \tag{1}$$

is a path-decomposition of G. We first show that each vertex v is in some bag. If v is an end-vertex of some edge e_i, then $v \in V_i$. Otherwise $v = v_{i,j}$ for some i, j, implying that $v \in V_{i,j}$, as desired. We now show that each vertex v is in a sequence of consecutive bags. Suppose that $v \in V_i \cap V_p$ and

$i < j < p$. Thus $e_i \prec e_j \prec e_p$. Our goal is to show that $v \in V_j$. If v is an end-vertex of e_j, then $v \in V_j$. So we may assume that v is not an end-vertex of e_j. By symmetry, we may assume that v is to the left of the end-vertex of e_j that is in the same layer as v. Thus, v is not an end-vertex of e_p. Since $v \in V_p$, there is an arc $\overrightarrow{yv}$ that crosses e_p or such that y is an end-vertex of e_p. Since $e_j \prec e_p$, this arc $\overrightarrow{yv}$ crosses e_j. Thus $v \in V_j$, as desired. This shows that v is in a (possibly empty) sequence of consecutive bags $V_i, V_{i+1}, \dots, V_j$. If $v \in V_i$ then $v \in V_{i,j}$ for all $j \in \{1, \dots, m_i\}$, and $v \in V_{i-1,j}$ for all $j \in \{1, \dots, m_{i-1}\}$. It remains to consider the case in which v is in no set V_i. Since the end-vertices of e_i are in V_i, we have that $v = v_{i,j}$ for some i, j. Since Y_i is an independent set, v is adjacent to no other vertex in Y_i. Moreover, if there is an arc $\overrightarrow{zv}$ in G, then either z is an end-vertex of e_i or e_{i-1}, or $\overrightarrow{zv}$ crosses e_{i-1} or e_i, implying v is in $V_{i-1} \cup V_i$, which is not the case. Hence v has indegree 0, implying $V_{i,j}$ is the only bag containing v. This completes the proof that v is in a sequence of consecutive bags in (1). Finally, we show that the end-vertices of each edge are in some bag. Consider an arc $\overrightarrow{vw}$ in G. If $v = v_{i,j}$ for some i, j, then $v, w \in V_{i,j}$, as desired. Otherwise, v is an end-vertex of some e_i, implying $v, w \in V_i$, as desired. Hence the sequence in (1) defines a path-decomposition of G.

We now bound the width of this path-decomposition. The goal is to identify certain subgraphs of G to which Lemma 5 is applicable.

As illustrated in Figure 7, for $i, j \in \{0, 1, \dots, n\}$, let $Y_{i,j}$ be the set of vertices $v \in Y_i$ such that there is an arc $\overrightarrow{zv}$ in G with $z \in Y_j$. Suppose that $|Y_{i,j}| \geqslant 2k^2|j - i| + 1$ for some $i, j \in \{0, 1, \dots, n\}$. Since Y_i is an independent set, $i \neq j$. Without loss of generality, $i < j$ and there exists $Z \subseteq Y_{i,j} \cap A$ with $|Z| \geqslant k^2(j - i) + 1$. Let H_1 be the subgraph of G consisting of all arcs $\overrightarrow{zv}$ in G with $z \in Y_j \cap B$ and $v \in Z$ (and their end-vertices). If H_1 has a non-crossing $(j - i + 1)$-matching M, then $(X \setminus \{e_{i+1}, \dots, e_j\}) \cup M$ is a non-crossing matching in G larger than X, thus contradicting the choice of X. Hence H_1 has no non-crossing $(j - i + 1)$-matching. By construction, H_1 has no $(k + 1)$-crossing, every vertex in $V(H_1) \cap A$ has degree at least 1 in H_1, and every vertex in $V(H_1) \cap B$ has degree at most k in H_1. By Lemma 5 applied to H_1 with $\ell = j - i$ and $d = k$, we have $|Z| = |V(H_1) \cap A| \leqslant k^2(j - i)$, which is a contradiction. Hence $|Y_{i,j}| \leqslant 2k^2|j - i|$ for all $i, j \in \{0, 1, \dots, n\}$.

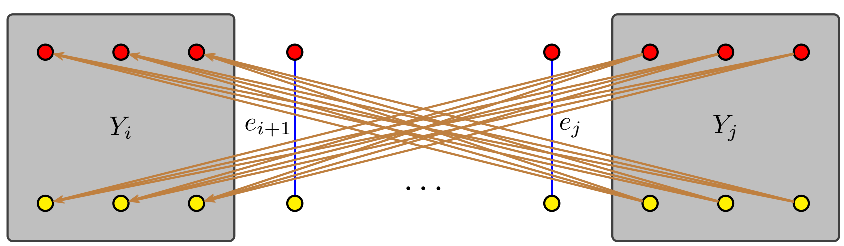

Figure 7: If many vertices in Y_i are the head of an arc starting in Y_j, then there is a large non-crossing matching amongst these edges, which can replace $e_{i+1}, \dots, e_j$ in M, contradicting the maximality of M.

This bound on $|Y_{i,j}|$ is useful if $|i - j|$ is 'small', but not useful if $|i - j|$ is 'big'. We now deal with this case.

As illustrated in Figure 8, for $i \in \{1, \dots, n\}$, let P_i be the set of vertices v in G for which there is an arc $\overrightarrow{zv}$ in G that crosses $e_{i-s+1}, e_{i-s+2}, \dots, e_i$ or crosses $e_i, e_{i+1}, \dots, e_{i+s-1}$. Suppose that $P_i \geqslant 4k^2(t - 1) + 1$. Without loss of generality, there exists $Q \subseteq P_i \cap A$ with $|Q| \geqslant k^2(t - 1) + 1$ such that for each vertex $v \in Q$ there is an arc $\overrightarrow{zv}$ in G that crosses $e_i, e_{i+1}, \dots, e_{i+s-1}$. Let H_2

be the subgraph of G consisting of all such arcs and their end-vertices. So $V(H_2) \cap A = Q$. If H_2 has a non-crossing t-matching M, then $(\{e_i, e_{i+1}, \dots, e_{i+s-1}\}, M)$ is an (s,t)-crossing. Thus H_2 has no non-crossing t-matching. By construction, H_2 has no $(k+1)$-crossing, every vertex in $V(H_2) \cap A$ has degree at least 1 in H_2, and every vertex in $V(H_2) \cap B$ has degree at most k in H_2. By Lemma 5 applied to H_2 with $\ell = t-1$ and $d = k$, we have $|Q| = |V(H_2) \cap A| \leqslant k^2(t-1)$, which is a contradiction. Hence $|P_i| \leqslant 4k^2(t-1)$ for all $i \in \{1, \dots, n\}$.

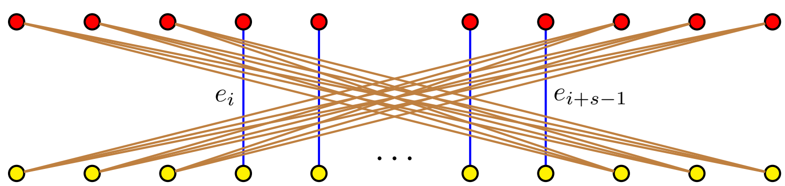

Figure 8: If many vertices are the head of an arc crossing $e_i, e_{i+1}, \dots, e_{i+s-1}$, then amongst these edges there is a non-crossing t-matching, implying that G has an (s,t)-crossing, which is a contradiction.

Consider a bag V_i, which consists of N_i along with every vertex $v \in V(G)$ such that some arc $\overrightarrow{zv} \in E(G)$ crosses e_i. Thus

$$\begin{aligned}
|V_i| &= |N_i| + |P_i| + \sum_{a,b \in \{0,1,\dots,s-2\}} |Y_{i-a,i+b}| \\
&\leqslant 2(k+1) + 4k^2(t-1) + \sum_{a,b \in \{0,1,\dots,s-2\}} 2k^2 |(i+b) - (i-a)| \\
&= 2(k+1) + 4k^2(t-1) + 2k^2 \sum_{a,b \in \{0,1,\dots,s-2\}} (a+b) \\
&= 2(k+1) + 4k^2(t-1) + 2k^2 \left((s-1) \left(\sum_{a \in \{0,1,\dots,s-2\}} a \right) + (s-1) \left(\sum_{b \in \{0,1,\dots,s-2\}} b \right) \right) \\
&= 2(k+1) + 4k^2(t-1) + 2k^2(s-1)^2(s-2).
\end{aligned}$$

Hence

$$\begin{aligned}
|V_{i,j}| \leqslant |V_i| + |V_{i+1}| + (k+1) &\leqslant 4(k+1) + 8k^2(t-1) + 4k^2(s-1)^2(s-2) + (k+1) \\
&\leqslant 8k^2(t-1) + 4k^2(s-1)^2(s-2) + 5(k+1).
\end{aligned}$$

Therefore the path-decomposition of G defined in (1) has width at most $8k^2(t-1) + 4k^2(s-1)^2(s-2) + 5k + 4$. □

Proof of Theorem 3: Let G be a bipartite graph with pathwidth at most k. Our goal is to construct a 2-layer drawing of G with no $(k+2)$-crossing and no $(k+1,k+1)$-crossing. Let $(X_1, \dots, X_n)$ be a path-decomposition of G with width k. Let $\ell(v) := \min\{i : v \in X_i\}$ and $r(v) := \max\{i : v \in X_i\}$ for each $v \in V(G)$. We may assume that $\ell(v) \neq \ell(w)$ for all distinct $v, w \in V(G)$. Let $\{A, B\}$ be a bipartition of G. Consider the 2-layer drawing of G, in which each $v \in A$ is at $(\ell(v), 0)$, each $v \in B$ is at $(\ell(v), 1)$, and each edge is straight.

As illustrated in Figure 9, suppose that $\{v_1w_1, \dots, v_{k+2}w_{k+2}\}$ is a $(k+2)$-crossing in this drawing, where $v_i \in A$ and $w_i \in B$.

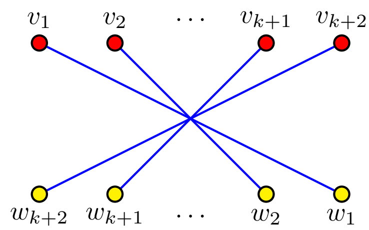

Figure 9: A $(k+2)$-crossing.

Without loss of generality,

$$\ell(v_1) < \ell(v_2) < \cdots < \ell(v_{k+2}) \quad \text{and} \quad \ell(w_{k+2}) < \ell(w_{k+1}) < \cdots < \ell(w_1). \tag{2}$$

For each $i \in \{1, \dots, k+2\}$, if $\ell(v_i) < \ell(w_i)$ then let $I_i := \{\ell(v_i), \dots, \ell(w_i)\}$; otherwise let $I_i := \{\ell(w_i), \dots, \ell(v_i)\}$. By (2), $I_i \cap I_j \neq \varnothing$ for distinct $i, j \in \{1, \dots, k+2\}$. By the Helly property for intervals, there exists $p \in I_1 \cap \cdots \cap I_{k+2}$. Thus v_i or w_i is in X_p for each $i \in \{1, \dots, k+2\}$. Hence $|X_p| \geqslant k+2$, which is a contradiction. Therefore there is no $(k+2)$-crossing.

As illustrated in Figure 10, consider an (s,s)-crossing $(\{v_1w_1, \dots, v_sw_s\}, \{x_1y_1, \dots, x_sy_s\})$ in this drawing, where $v_i, x_i \in A$ and $w_i, y_i \in B$.

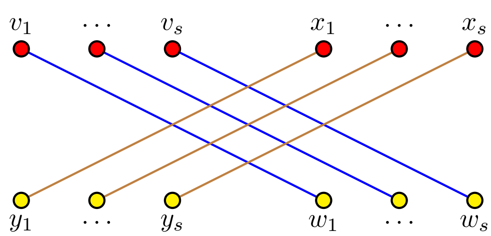

Figure 10: An (s,s)-crossing.

Without loss of generality,

$$\ell(v_1) < \cdots < \ell(v_s) < \ell(x_1) < \cdots < \ell(x_s) \quad \text{and}$$
$$\ell(y_1) < \cdots < \ell(y_s) < \ell(w_1) < \cdots < \ell(w_s).$$

We claim that $s \leqslant k$. If $\ell(v_s) < \ell(w_1)$ then $\ell(v_1) < \cdots < \ell(v_s) < \ell(w_1) < \cdots < \ell(w_s)$, implying $v_1, \dots, v_s, w_1 \in X_{\ell(w_1)}$, and $s+1 \leqslant |X_{\ell(w_1)}| \leqslant k+1$, as desired. If $\ell(y_s) < \ell(x_1)$ then $\ell(y_1) < \cdots < \ell(y_s) < \ell(x_1) < \cdots < \ell(x_s)$, implying $y_1, \dots, y_s, x_1 \in X_{\ell(x_1)}$, and $s+1 \leqslant |X_{\ell(x_1)}| \leqslant k+1$, as desired. Now assume that $\ell(w_1) < \ell(v_s)$ and $\ell(x_1) < \ell(y_s)$. Thus $\ell(w_1) < \ell(v_s) < \ell(x_1) < \ell(y_s)$, which is a contradiction since $\ell(y_s) < \ell(w_1)$. Hence $s \leqslant k$ and the drawing of G has no $(k+1, k+1)$-crossing. □

References

[1] P. Angelini, G. Da Lozzo, H. Förster, and T. Schneck. 2-layer k-planar graphs - density, crossing lemma, relationships, and pathwidth. In D. Auber and P. Valtr, editors, *Graph Drawing and Network Visualization - 28th International Symposium, GD 2020*, volume 12590 of *Lecture Notes in Computer Science*, pages 403–419. Springer, 2020. doi:10.1007/978-3-030-68766-3_32.

[2] M. J. Bannister, W. E. Devanny, V. Dujmović, D. Eppstein, and D. R. Wood. Track layouts, layered path decompositions, and leveled planarity. *Algorithmica*, 81(4):1561–1583, 2019. doi:10.1007/S00453-018-0487-5.

[3] T. Biedl, M. Chimani, M. Derka, and P. Mutzel. Crossing number for graphs with bounded pathwidth. *Algorithmica*, 82(2):355–384, 2020. doi:10.1007/S00453-019-00653-X.

[4] D. Bienstock, N. Robertson, P. D. Seymour, and R. Thomas. Quickly excluding a forest. *J. Comb. Theory, Ser. B*, 52(2):274–283, 1991. doi:10.1016/0095-8956(91)90068-U.

[5] H. L. Bodlaender. A partial k-arboretum of graphs with bounded treewidth. *Theor. Comput. Sci.*, 209(1-2):1–45, 1998. doi:10.1016/S0304-3975(97)00228-4.

[6] S. Cornelsen, T. Schank, and D. Wagner. Drawing graphs on two and three lines. *J. Graph Algorithms Appl.*, 8(2):161–177, 2004. doi:10.7155/JGAA.00087.

[7] E. Di Giacomo, W. Didimo, P. Eades, and G. Liotta. 2-layer right angle crossing drawings. *Algorithmica*, 68(4):954–997, 2014. doi:10.1007/S00453-012-9706-7.

[8] R. Diestel. Graph minors 1: A short proof of the path-width theorem. *Comb. Probab. Comput.*, 4:27–30, 1995. doi:10.1017/S0963548300001450.

[9] R. P. Dilworth. A decomposition theorem for partially ordered sets. *Annals of Mathematics*, 51(1):161–166, 1950. doi:10.2307/1969503.

[10] V. Dujmović, M. R. Fellows, M. T. Hallett, M. Kitching, G. Liotta, C. McCartin, N. Nishimura, P. Ragde, F. A. Rosamond, M. Suderman, S. Whitesides, and D. R. Wood. A fixed-parameter approach to 2-layer planarization. *Algorithmica*, 45(2):159–182, 2006. doi:10.1007/S00453-005-1181-Y.

[11] V. Dujmović, M. R. Fellows, M. Kitching, G. Liotta, C. McCartin, N. Nishimura, P. Ragde, F. A. Rosamond, S. Whitesides, and D. R. Wood. On the parameterized complexity of layered graph drawing. *Algorithmica*, 52(2):267–292, 2008. doi:10.1007/S00453-007-9151-1.

[12] V. Dujmović, P. Morin, and D. R. Wood. Layout of graphs with bounded tree-width. *SIAM J. Comput.*, 34(3):553–579, 2005. doi:10.1137/S0097539702416141.

[13] V. Dujmovic, P. Morin, and C. Yelle. Two results on layered pathwidth and linear layouts. *J. Graph Algorithms Appl.*, 25(1):43–57, 2021. doi:10.7155/JGAA.00549.

[14] V. Dujmović, A. Pór, and D. R. Wood. Track layouts of graphs. *Discret. Math. Theor. Comput. Sci.*, 6(2):497–522, 2004. doi:10.46298/DMTCS.315.

[15] V. Dujmović and S. Whitesides. An efficient fixed parameter tractable algorithm for 1-sided crossing minimization. *Algorithmica*, 40(1):15–31, 2004. doi:10.1007/S00453-004-1093-2.

[16] P. Eades and S. Whitesides. Drawing graphs in two layers. *Theor. Comput. Sci.*, 131(2):361–374, 1994. doi:10.1016/0304-3975(94)90179-1.

[17] P. Eades and N. C. Wormald. Edge crossings in drawings of bipartite graphs. *Algorithmica*, 11(4):379–403, 1994. doi:10.1007/BF01187020.

[18] S. Felsner, G. Liotta, and S. K. Wismath. Straight-line drawings on restricted integer grids in two and three dimensions. *J. Graph Algorithms Appl.*, 7(4):363–398, 2003. doi:10.7155/JGAA.00075.

[19] F. Harary and A. Schwenk. A new crossing number for bipartite graphs. 1:203–209.

[20] P. Hlinený. Crossing-number critical graphs have bounded path-width. *J. Comb. Theory, Ser. B*, 88(2):347–367, 2003. doi:10.1016/S0095-8956(03)00037-6.

[21] H. Nagamochi. An improved bound on the one-sided minimum crossing number in two-layered drawings. *Discret. Comput. Geom.*, 33(4):569–591, 2005. doi:10.1007/S00454-005-1168-0.

[22] H. Nagamochi. On the one-sided crossing minimization in a bipartite graph with large degrees. *Theor. Comput. Sci.*, 332(1-3):417–446, 2005. doi:10.1016/J.TCS.2004.10.042.

[23] N. Robertson and P. D. Seymour. Graph minors. i. excluding a forest. *J. Comb. Theory, Ser. B*, 35(1):39–61, 1983. doi:10.1016/0095-8956(83)90079-5.

[24] M. Suderman. Pathwidth and layered drawings of trees. *Int. J. Comput. Geom. Appl.*, 14(3):203–225, 2004. doi:10.1142/S0218195904001433.